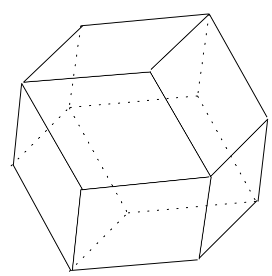
SPECIAL K
Saturday November 2, 1996
9:00 am - 12:00 pm

1: Prove that $\dfrac{1}{\log_2 1996} + \dfrac{1}{\log_3 1996} + \cdots + \dfrac{1}{\log_{1996} 1996} = \dfrac{1}{\log_{1996!} 1996}$.

2: Suppose that a, b and c are positive integers with no factor common to all three. Furthermore, suppose that $a^{-1} + b^{-1} = c^{-1}$. Prove that $a+b$, $a-b$ and $b-c$ are all perfect squares.

3: A polyhedron called a rhombic dodecahedron is shown in the figure below. It has 12 congruent faces, each a rhombus. Suppose a beetle travels around this polyhedron moving from vertex to vertex along edges that join neighbouring vertices. Starting from any vertex, is it possible for the beetle to visit all of the vertices without returning to any vertex previously visited?



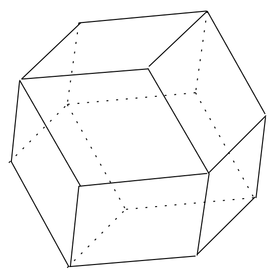
4: Suppose that the function $f : \mathbf{R} \to \mathbf{R}$ is twice continuously differenttiable, and that $\big|f(x)\big| \le 1$ for all $x \in \mathbf{R}$. If $f(0)^2 + f'(0)^2 = 4$, prove that there exists a real number y such that $f(y) + f''(y) = 0$.

5: Let p be an odd prime, and let $k = 1 \pmod{p}$. Prove that for every positve integer n, the highest power of p dividing n is equal to the highest power of p dividing $1+k+k^2+\cdots+k^{n-1}$.

BIG E
Saturday November 2, 1996
9:00 am - 12:00 pm

1: Prove that $\dfrac{\sin x}{\sin x} + \dfrac{\sin 3x}{\sin x} + \dfrac{\sin 5x}{\sin x} + \cdots + \dfrac{\sin(2n-1)x}{\sin x} = \left(\dfrac{\sin nx}{\sin x}\right)^2$.

2: A polyhedron called a rhombic dodecahedron is shown in the figure below. It has 12 congruent faces, each a rhombus. Suppose a beetle travels around this polyhedron moving from vertex to vertex along edges that join neighbouring vertices. Starting from any vertex, is it possible for the beetle to visit all of the vertices without returning to any vertex previously visited?



3: Let a and b be two independent random numbers drawn uniformly from the open interval $(0,1)$. Let y be the unique root of the polynomial $ax^3 + bx - 1$. Find the median of the distribution of y, that is find the value t such that $P(y \leq t) = \frac{1}{2}$.

4: Prove that for any positive integer n and any odd positive integer k,

$$1^k + 2^k + \cdots + n^k$$

is divisible by $1 + 2 + \cdots + n$.

5: Let $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$. Prove that

$$\det \begin{pmatrix} \exp(x_1 y_1) & \exp(x_1 y_2) & \cdots & \exp(x_1 y_n) \\ \exp(x_2 y_1) & \exp(x_2 y_2) & \cdots & \exp(x_2 y_n) \\ \exp(x_3 y_1) & \exp(x_3 y_2) & \cdots & \exp(x_3 y_n) \\ \vdots & \vdots & & \vdots \\ \exp(x_n y_1) & \exp(x_n y_2) & \cdots & \exp(x_n y_n) \end{pmatrix} > 0\,.$$