SPECIAL K
Saturday November 5, 1994
9:00 am - 12:00 noon

1: Does there exists an infinite sequence of positive integers $a_1, a_2, a_3, \cdots$ such that each of the numbers ${a_1}^2 + {a_2}^2 + \cdots + {a_n}^2$ is a perfect square for all $n = 1, 2, 3, \cdots$? Justify your answer.

2: Prove that the famous four colour map theorem does not work on a Möbius band by dividing the Möbius band into five regions so that any two of them have a common boundary. (In dividing and colouring the Möbius band remember that it has no thickness).

3: Let M be the 7×7 matrix of positive integers whose $(j,k)^{th}$ entry is the greatest common divisor of j and k. Evaluate the determinant of M. Generalize your work to the $n \times n$ case.

4: Let n be a positive integer greater than 2, and let f be any polynomial of degree not exceeding $n-2$. If $a_1, a_2, \cdots, a_n$ are any distinct real numbers, and $p(t) = (t-a_1)(t-a_2)\cdots(t-a_n)$, prove that

$$\sum_{j=1}^{n} \frac{f(a_j)}{p'(a_j)} = 0\,.$$

5: Determine all solutions in integers x, y, z to the equation

$$\sqrt[3]{x+\sqrt{y}} + \sqrt[3]{x-\sqrt{y}} = z\,.$$

BIG E
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1: For any positive integer n, we define the positive integer $s(n)$ to be the smallest positive integer for which

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{s(n)} \geq n \,.$$

Evaluate

$$\lim_{n\to\infty} \frac{s(n+1)}{s(n)} \,.$$

2: Prove that the famous four colour map theorem does not work on a Möbius band by dividing the Möbius band into five regions so that any two of them have a common boundary. (In dividing and colouring the Möbius band remember that it has no thickness).

3: For any $n = 1, 2, \cdots$, let M be the $n \times n$ matrix of positive integers whose $(j,k)^{th}$ entry is the greatest common divisor of j and k. Evaluate the determinant of M.

4: Let $x \wedge y$ denote the smaller of the numbers x and y. For what values of $\lambda > 0$ does the equation

$$\int_0^1 (x \wedge y) f(y)\, dy = \lambda\, f(x)$$

have continuous solutions f which do not vanish identically on $(0,1)$. Find these solutions.

5: Suppose h is a bounded strictly decreasing real-valued function with $h(\theta^*) = 0$ for some $\theta^* \in \mathbf{R}$. For $n \geq 0$ define

$$\theta_{n+1} = \theta_n + \frac{1}{n+1}\, h(\theta_n) \,.$$

Show that for any choice of θ_0 in $\mathbf{R}$ we have $\lim_{n\to\infty} \theta_n = \theta^*$.