SPECIAL K
Saturday November 13, 1993
9:00 am - 12:00 pm

1: Evaluate

$$\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\, dx\,.$$

2: Find all solutions in nonnegative integers x and y to the equation

$$x\,(x+1)(x+2)(x+3) = \sum_{i=0}^{y}(2i+1)\,.$$

3: Consider a set of n points placed arbitrarily in d-dimensional Euclidean space $\mathbf{R}^d$. From each point, we draw an arrow to its nearest neighbour among the $n-1$ other points (using the usual Euclidean measure of distance). We assume that each point has a unique nearest neighbour. Two of the n points are said to be *connected* if there is an arrow from one to the other. Two points are said to lie in the same *cluster* if there is a path of connected points from one to the other. Show that each cluster has exactly one *reflexive* pair of points, i.e. a pair of points joined by arrows in both directions.

4: On a blackboard are written the numbers

$$1\,,\ \tfrac{1}{2}\,,\ \tfrac{1}{3}\,,\ \tfrac{1}{4}\,,\ \cdots\,,\ \tfrac{1}{100}\,.$$

At each step, two numbers a and b are selected arbitrarily from the list, deleted, and replaced by the single number $a+b+ab$. After 99 steps, one number is left. What are the possible values of this number?

5: Two points, A and B, are placed in the interior of a circle. The point C is chosen to lie on the boundary of the circle such that the angle ACB is maximized. Find general conditions under which the point C is unique, and construct a method for finding C under these conditions.

BIG E
Saturday November 13, 1993
9:00 am - 12:00 pm

1: Evaluate

$$\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\,dx\,.$$

2: Let f be a continuous function of the closed interval $[0,1]$ into itself. The function f is said to be an *involution* if for every point $x \in [0,1]$ we have $f\big(f(x)\big) = x$. The involution

$$f(x) = x$$

is said to be *trivial*. Prove that every non-trivial involution has exactly one fixed point, i.e. one point y with $f(y) = y$.

3: Let $x_1, x_2, x_3, \cdots$ be a sequence of positive real numbers. Define

$$\overline{x}_n = \frac{x_1 + x_2 + \cdots + x_n}{n}\,.$$

Show that if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x_n} < \infty$ then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\overline{x}_n} < \infty$.

4: On a blackboard are written the numbers

$$1\ ,\ \tfrac{1}{2}\ ,\ \tfrac{1}{3}\ ,\ \tfrac{1}{4}\ ,\ \cdots\ ,\ \tfrac{1}{100}\,.$$

At each step, two numbers a and b are selected arbitrarily from the list, deleted, and replaced by the single number $a+b+ab$. After 99 steps, one number is left. What are the possible values of this number?

5: A black and white cat walks on the plane which is divided into cells by a square grid. Each cell is either black or white. Initially, the cat is sitting on a cell, call it the origin, heading in one of the four compass directions. It proceeds to walk from cell to cell according to the following rule: it moves one cell over in the direction it is heading, when iands on a white (black) cell it rotates its heading 90 degrees to the right (left) and paints the cell the opposite colour with a brush attached to its tail. Prove that the cat's trajectory is unbounded.