SPECIAL K
Saturday November 7, 1992
9:00 am - 12:00 noon

1: Let $a_1, a_2, a_3 \cdots, a_n$ be a rearrangement of the numbers $1, 2, 3, \cdots, n$ where n is odd. Prove that that

$$(a_1 - 1)(a_2 - 1)(a_3 - 1) \cdots (a_n - 1)$$

is an even number.

2: Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2} F_{n+3}}$$

where F_n is the n^{th} Fibonacci number with $F_1 = 1$ and $F_2 = 1$.

3: Let $S_1, S_2, \cdots, S_n$ be n circles with common centre O in the plane. We suppose that S_k has radius k. A random point A is uniformly distributed over the region bounded by the circle S_n. Triangle ABC is an equilateral triangle having centre O. Find the probability that ABC cuts exactly m of the circles $S_1, S_2, \cdots, S_n$.

4: Let two opposite sides of a cyclic quadrilateral be extended to meet in P, and the other two sides to meet in Q. Show that the internal bisectors of the angles at P and Q are perpendicular.

5: A number of gas stations are placed along a road which forms a circle. The total amount of gas in all the stations is sufficient to allow a car to drive exactly once around the circular road but no further. Is it possible to configure a number of gas stations and to allocate the gas to those stations so that no matter where the car starts from it is unable to make a full clockwise circle of the road? Assume that the car starts with an empty tank at some gas station.

BIG E
Saturday November 7, 1992
9:00 am - 12:00 noon

1: Let $f(n)$ be the n-dimensional volume of an n-dimensional ball of unit radius. Thus $f(1) = 2$, $f(2) = \pi$ and $f(3) = \frac{4\pi}{3}$, etc. Find the value of n which maximizes $f(n)$.

2: Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2} F_{n+3}}$$

where F_n is the n^{th} Fibonacci number with $F_1 = 1$ and $F_2 = 1$.

3: Let $f : \mathbf{R} \to \mathbf{R}$ be a twice differentiable function. Suppose that

$$f(x)^2 + f'(x)^2 + f''(x)^2 \geq 1\,.$$

Prove that f has only finitely many roots on any finite interval (a, b).

4: Let f be a function of two real variables satisfying $f(x, 0) = x$ and $f(x, y) = f(y, x) = -f(-y, -x)$ for all real x and y. Suppose that the periodic sequence $x_0, x_1, x_2, \cdots$ satisfies

$$\begin{array}{ll} (i) & x_{n+1} = f(x_{n-1}, x_n) \\ (ii) & x_{n-1} = f(x_{n+1}, -x_n)\,. \end{array}$$

In addition, suppose that, for some positive integer d, $x_n = 0$ if and only if n is a multiple of d. Show that if $d > 3$ the period of the sequence is even.

5: Three distinct points A, B and C are randomly scattered in the plane. We assume that they are independent and identically distributed. Let p be the probability that triangle ABC has an interior angle of at least 120 degrees. Prove that $p \geq \frac{1}{20}$.