SPECIAL K
Saturday November 2, 1991
9:00 am - 12:00 noon

1: Show that if a real valued function f verifies

$$f(x+y) = f(xy)$$

for all (strictly) positive real numbers x and y, then f is constant over $(0, \infty)$.

2: A list is made of all subsets of the set $S = \{1, 2, \cdots, n\}$, including S and the empty set in the list. Subsets $A_1, A_2, \cdots, A_r$, $r > 1$, are chosen at random from the list (a subset can be chosen more than once). Find the probability that the chosen subsets are pairwise disjoint (i.e. $A_i \cap A_j = \emptyset$ for all $1 \leq i < j \leq r$).

3: Let P be a polynomial on the real numbers. Show that

$$\left| P(k) - 3^k \right| < 1 \text{ for all } k = 0, 1, 2, \cdots, n$$

implies that the degree of P is not less than n.

4: A set of $2n+2$ 2-vectors $V_1, V_2, \cdots, V_{2n+2}$ is made by selecting the entries arbitrarily from the set $\{1, 2, 4, 8, \cdots, 2^n\}$. Show that there exists a pair of vectors V_i, V_j, $i \neq j$, such that the 2×2 matrix

$$A_{ij} = \begin{pmatrix} V_i \\ V_j \end{pmatrix}$$

has determinant zero.

5: Let $\triangle ABC$ be any triangle and A', B', C' points on sides BC, CA and AB, respectively, such that the circles inscribed in triangles $\triangle AC'B'$, $\triangle BA'C'$, $\triangle CB'A'$ have equal radii r. Let $\overline{r}$ be the radius of the circle inscribed in $\triangle A'B'C'$ and R that of the circle inscribed in $\triangle ABC$. Prove that

$$R = r + \overline{r}\,.$$

BIG E
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1: A list is made of all subsets of the set $S = \{1, 2, \cdots, n\}$, including S and the empty set in the list. Subsets $A_1, A_2, \cdots, A_r$, $r > 1$, are chosen at random from the list (a subset can be chosen more than once). Find the probability that the chosen subsets are pairwise disjoint (i.e. $A_i \cap A_j = \emptyset$ for all $1 \le i < j \le r$).

2: Let P be a polynomial on the real numbers. Show that

$$\left|P(k) - 3^k\right| < 1 \text{ for all } k = 0, 1, 2, \cdots, n$$

implies that the degree of P is not less than n.

3: A set K in the plane is said to be *Valentine convex* if for any set $\{x, y, z\}$ of points in K one of the three line segments xy, yz, zx is contained in K.

(a) Show that the union of any two convex sets is Valentine convex. Show that there exist three convex sets whose union is not Valentine convex.

(b) Give an example of a Valentine convex set which cannot be expressed as the union of two convex sets.

4: A *figure eight* is a closed curve that intersects itself exactly once. Show that any collection of disjoint figure eights in the plane must necessarily be countable.

5: Let $f : \mathbf{R}^+ \to \mathbf{R}$ be twice continuously differentiable. such that

(a) $f(0) = f'(0) = 0$,

(b) $0 \le 3\, f''(x) \le \sqrt{1 + (f'(x))^2}\,\big(\cos(f(x)) + 2\big)$ for all $x > 0$.

Prove that $0 \le f(x) \le \cosh x - 1$ for all $x > 0$.