Bernoulli Trial 45^2

1. There are 15 questions. Answer T or F only.
2. Put your name, your BT ID, and the question number on each answer slip.
3. Correct answer = 1 point; Incorrect answer = -1 point; No answer = 0 point.
4. At least 12 questions need to be answered to qualify for the prizes.
5. Prizes:
    - First place: $200
    - Second place: $100
    - Third place: $50
    - Last place: $200

1: (2 minutes)

T/F: There are infinitely many positive integers d, n such that $d \mid n$ and

$$\binom{2d}{d} \nmid \binom{2n}{n}.$$

1: (2 minutes)

T/F: There are infinitely many positive integers d, n such that $d \mid n$ and

$$\binom{2d}{d} \nmid \binom{2n}{n}.$$

T: If p is a prime between d and $2d$ and between $2n/3$ and n, then it will divide $\binom{2d}{d}$ but not $\binom{2n}{n}$. For example, $5 \mid \binom{6}{3}$ but $5 \nmid \binom{12}{6}$. We take $n = 2d$. Then we need there to be a prime between $4d/3$ and $2d$. Such a prime exists for any $d \geq 3$.

Gian says: Take $d = 2$ and $n = (9^k - 1)/2$ so that $d \mid n$. For any $j \leq 2k - 1$, the remainder $2n\%3^j = 2$ and $n\%3^j = 1$ so $3 \nmid \binom{2n}{n}$. More generally, fix any $d \geq 2$, let p be a prime between d and $2d$. Then $p \mid \binom{2d}{d}$ and is coprime to d. Take

$$n = (p^{o_{2d}(p)k} - 1)/2.$$

2: (3 minutes)

T/F: $x^{46} + 69x + 2025$ is irreducible in $\mathbb{Z}[x]$.

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T: This smells like Eisenstein's criterion but

$$f(x) := x^{46} + 69x + 2025 \equiv x^{46} + 1 \equiv (x^2 + 1)^{23} \pmod{23}.$$

The key point here is that $23 \equiv 3 \pmod 4$ so that $x^2 + 1$ is irreducible in $\mathbb{F}_{23}[x]$. We can write $f(x) = (x^2 + 1)^{23} + 23g(x)$ where

$$g(x) = -x^{44} - \frac{1}{23}\binom{23}{2}x^{42} \cdots - x^2 + 3x + 88.$$

Suppose $f(x) = h(x)k(x)$ where $h, k \in \mathbb{Z}[x]$ are monic of degree at least 1. Then $\bar{h}\bar{k} = (x^2 + 1)^{23}$. Since $x^2 + 1$ is irreducible, we have $\bar{h} = (x^2 + 1)^a$ and $\bar{k} = (x^2 + 1)^b$ for some positive a, b with $a + b = 23$. Write $h = (x^2 + 1)^a + 23h_1(x)$ and $k = (x^2 + 1)^b + 23k_1(x)$. We see that

$$g(x) = (x^2 + 1)^a k_1(x) + (x^2 + 1)^b h_1(x) + 23h_1(x)k_1(x).$$

So $x^2 + 1$ divides $\bar{g}$ in $\mathbb{F}_{23}[x]$. However, $g(x) \equiv 3x + \cdots \pmod{x^2 + 1}$. Contradiction.

Gian says: So essentially $f(x) \equiv u^n \pmod p$ and $f(x) \notin (p, u)^2$ where $p = 23$ and $u = x^2 + 1$ is irreducible mod p.

3: (3 minutes)

T/F: There is a unique digit $d = 1, \ldots, 9$ such that if 2^n and 5^n start with the same digit for some $n \in \mathbb{N}$, then that digit is d.

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T: Since $2^n \cdot 5^n = 10^n$, we need their product to begin with 10. This is only possible if $d = 3$.

4: (4 minutes)

T/F: For any odd prime number p,

$$\left(\frac{p-1}{2}\right)^3 \mid \sum_{o=1}^{p-1}\sum_{r=1}^{p-1}\sum_{z=1}^{p-1}\left\lfloor \frac{orz}{p} \right\rfloor.$$

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T: Since p is a prime, the remainder of orz mod p is never 0. We pair orz with $or(p-z)$ to see that

$$\left\lfloor\frac{orz}{p}\right\rfloor + \left\lfloor\frac{or(p-z)}{p}\right\rfloor = \frac{orz}{p} + \frac{or(p-z)}{p} - 1.$$

Hence

$$\begin{aligned}\sum_{o=1}^{p-1}\sum_{r=1}^{p-1}\sum_{z=1}^{p-1}\left\lfloor\frac{orz}{p}\right\rfloor &= \sum_{o=1}^{p-1}\sum_{r=1}^{p-1}\sum_{z=1}^{p-1}\left(\frac{orz}{p}\right) - \frac{(p-1)^3}{2}\\ &= \frac{1}{p}\left(\sum_{n=1}^{p-1} n\right)^3 - \frac{(p-1)^3}{2}\\ &= \frac{1}{p}\left(\frac{p(p-1)}{2}\right)^3 - \frac{(p-1)^3}{2},\end{aligned}$$

both terms are divisible by $((p-1)/2)^3$.

5: (4 minutes)

T/F: $\displaystyle\int_2^\infty \frac{1}{x^7 - x}\,dx > \frac{1}{365}$.

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F: For $x > 2$, we have $(x/2)^7 > (x/2)$ from which we get that

$$x^7 - x > \frac{126}{128}x^7.$$

Hence

$$\int_2^\infty \frac{1}{x^7 - x}\,dx < \frac{64}{63}\int_2^\infty \frac{1}{x^7}dx = \frac{1}{6}\frac{1}{63} = \frac{1}{378}.$$

In fact, we can compute this integral exactly:

$$\int_2^\infty \frac{1}{x^7 - x}\,dx = \int_2^\infty \frac{x^5}{x^{12} - x^6}\,dx = \frac{1}{6}\int_{64}^\infty \frac{1}{u^2 - u}\,du = \frac{1}{6}\int_{64}^\infty \frac{1}{u-1} - \frac{1}{u}\,du = \frac{1}{6}\ln(\frac{64}{63}).$$

6: (4 minutes)

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables uniform on $(0,1)$. Let

$$R_n = \sum_{k=1}^{n} \begin{cases} 1 & \text{if } X_k = \max\{X_1, \ldots, X_k\} \\ 0 & \text{otherwise} \end{cases}$$

For any random variable Y, let $E(Y)$ denote its expectation and $\text{Var}(Y) = E(Y^2) - E(Y)^2$ denots its variance.

T/F: $\lim_{n\to\infty} \Big(E(R_n) - \text{Var}(R_n)\Big) < \dfrac{\pi^2}{420/69}$.

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T/F: $\lim_{n\to\infty} \Big(E(R_n) - \mathrm{Var}(R_n) \Big) < \dfrac{\pi^2}{420/69}$.

F: Let

$$I_k = \begin{cases} 1 & \text{if } X_k = \max\{X_1, \ldots, X_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $E(I_k^2) = E(I_k) = 1/k$ and $E(I_k I_j) = 1/(kj)$ if $j < k$. By linearity, we have

$$\begin{aligned} E(R_n) &= \sum_{k=1}^{n} E(I_k) &= \sum_{k=1}^{n} \frac{1}{k} \\ E(R_n^2) &= \sum_{k=1}^{n} \sum_{j=1}^{n} E(I_k I_j) &= \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{1}{kj} + \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k^2} \right) &= E(R_n)^2 + E(R_n) - \sum_{k=1}^{n} \frac{1}{k^2}. \end{aligned}$$

Therefore,

$$E(R_n) - \mathrm{Var}(R_n) = \sum_{k=1}^{n} \frac{1}{k^2}.$$

The desired limit is $\dfrac{\pi^2}{6}$ which is more than $\dfrac{\pi^2}{420/69}$ as $420/69 > 6$.

7: (5 minutes)

T/F: For any integers $n, d \geq 3$, there exists a set $S \subseteq \{1, 2, \ldots, (2n)^d\}$ of size at least n^{d-2}/d that does not contain any 3-term arithmetic progression (i.e. there does not exist $a, b, c \in S$ such that $a + b = 2c$).

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T: Consider the set $A = [0, n-1]^d \cap \mathbb{Z}^d$ and the spheres $S_t : x_1^2 + \cdots + x_d^2 = t$ for $t = 0, \ldots, dn^2 - 1$. Every point of A lies on exactly one of these spheres. By the Pigeonhole principle, at least one of these spheres S_{t_0} contains at least $n^d/(dn^2)$ points of A. There are no arithmetic progressions among these points in $\mathbb{Z}^d$. Consider now the function $f : A \to \mathbb{Z}$ given by

$$f(x_1, \ldots, x_d) = x_1 + x_2(2n) + x_3(2n)^2 + \cdots + x_d(2n)^{d-1}.$$

Since all the x_i are bounded by n, we see that if $\mathbf{x}, \mathbf{y}, \mathbf{z} \in A$ satisfy $\mathbf{x} + \mathbf{y} \neq \mathbf{z} + \mathbf{z}$, then $f(\mathbf{x}) + f(\mathbf{y}) \neq f(\mathbf{z}) + f(\mathbf{z})$. So $f(S_{t_0} \cap A)$ also contains no arithmetic progressions. Furthermore, for any $\mathbf{x} \in A$, we have $f(\mathbf{x}) + 1 \in \{1, 2, \ldots, (2n)^d\}$.

With some optimizing, one obtains Behrend's result: there exists a subset of $\{1, 2, \ldots, N\}$ of size $Ne^{-C\sqrt{\log N}}$ with no 3-term arithmetic progression.

8: (2 minutes)

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ be four distinct points on (both branches of) the hyperbola $xy = 1$. Suppose they lie on a circle.

T/F: $x_1 x_2 x_3 x_4 = 1$.

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Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ be four distinct points on (both branches of) the hyperbola $xy = 1$. Suppose they lie on a circle.

T/F: $x_1x_2x_3x_4 = 1$.

T: Suppose the circle is given by $(x-a)^2 + (y-b)^2 = R^2$. Then x_1, x_2, x_3, x_4 are roots of

$$(x-a)^2 + (\frac{1}{x} - b)^2 = R^2.$$

This is secretly a quartic polynomial in x with constant term 1. So $x_1x_2x_3x_4 = 1$.

9: (3 minutes)

T/F: There exist **unique** bijections $f, g, h : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,

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T: We have

$$f(n)^3 - n^3 + (g(n)^3 + h(n)^3 + n^3 - 3ng(n)h(n)) = 0.$$

By AM-GM, since $g(n), h(n), n \geq 0$, we have

$$g(n)^3 + h(n)^3 + n^3 - 3ng(n)h(n) \geq 0$$

and so $f(n) \leq n$ for all $n \in \mathbb{N}$. Since f is a bijection, we see that $f(n) = n$ for all $n \in \mathbb{N}$. This also means that equality of AM-GM is satisfied and so $g(n) = h(n) = n$ for all $n \in \mathbb{N}$ as well.

10: (4 minutes)

For any positive integer n, let S_n denote the group of all permutations of $\{1, \ldots, n\}$. For each $\sigma \in S_n$, let $\operatorname{Orb}(\sigma)$ denote the number of cycles of σ (which is the same as the number of orbits as σ acts on $\{1, \ldots, n\}$).

T/F: $$\frac{1}{69!} \sum_{\sigma \in S_{69}} \operatorname{Orb}(\sigma) < 4.$$

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T/F: $\quad \dfrac{1}{69!} \displaystyle\sum_{\sigma \in S_{69}} \text{Orb}(\sigma) < 4.$

F: For each positive integer n, let X_n denote the number of cycles of a randomly chosen $\sigma \in S_n$. Then depending on if n is fixed, we have

$$E(X_n) = \frac{1}{n}(E(X_{n-1}) + 1) + \frac{n-1}{n} E(X_{n-1}) = E(X_{n-1}) + \frac{1}{n}.$$

Since $E(X_1) = 1$, we have

$$E(X_{69}) = \sum_{k=1}^{69} \frac{1}{k} > 4.$$

11: (3 minutes)

T/F: There does not exist $B \in M_{69\times 69}(\mathbb{R})$ such that $\dim_{\mathbb{R}}(\{BAB\colon A \in M_{69\times 69}(\mathbb{R})\}) = 2025$.

11: (3 minutes)

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F: Suppose $B \in M_{n\times n}(\mathbb{R})$ with rank r. Consider the linear map $T : A \mapsto BAB$. The kernel of T consists of $A \in M_{n\times n}(\mathbb{R})$ such that $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(\mathrm{Col}(B)) \subseteq \mathrm{Null}(B)$. Hence

$$\dim(\ker(T)) = (n-r)n + r(n-r).$$

Hence

$$\dim(\mathrm{im}(T)) = n^2 - 2r(n-r) = r^2.$$

Here, we simply take any B with rank 45.

This result can be used to prove that a group automorphism of $M_{n\times n}(k)$ preserves the rank of matrix.

12: (3 minutes)

T/F: There exists $a, b, c \in \mathbb{Z}$ such that $|\zeta_{13}^a + \zeta_{13}^b + \zeta_{13}^c + 1| = \sqrt{3}$, where $\zeta_{13} = e^{2\pi i/13}$.

12: (3 minutes)

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T: Squaring both sides, we find that we need

$$1 + \sum_{x \neq y \in \{a,b,c,0\}} \zeta_{13}^{x-y} = 0.$$

In other words, it is enough to find $a > b > c > 0$ such that $\{a-b, a-c, b-c, a, b, c\} = \{1, 2, 3, 4, 5, 6\}$. We take $a = 6$, $b = 5$, $c = 2$.

Note $a = 9$, $b = 3$, $c = 1$ also works as $\{a-b, a-c, b-c, a, b, c\} = \{1, 2, 3, 6, 8, 9\}$ has no repeats and their negatives cover the rest of the residue classes mod 13.

13: (4 minutes)

T/F: $\displaystyle\int_0^1 \frac{\sqrt{1+8x-8x^3}}{4x} - \sqrt{x^4-x+1} - \frac{1}{4x}\,dx \notin \mathbb{Q}$.

13: (4 minutes)

T/F: $\displaystyle\int_0^1 \frac{\sqrt{1+8x-8x^3}}{4x} - \sqrt{x^4-x+1} - \frac{1}{4x}\,dx \notin \mathbb{Q}$.

F: If we set

$$y = \frac{\sqrt{1+8x-8x^3}}{4x} - \frac{1}{4x} = \frac{-1+\sqrt{1-4\cdot(2x)(x^2-1)}}{2\cdot 2x},$$

then y satisfies the quadratic equation

$$2xy^2 + y + (x^2-1) = 0.$$

Note that as x goes from 0 to 1, y goes from 1 to 0. Viewing this as a quadratic equation in x, we have

$$x^2 + 2xy^2 + (y-1) = 0$$

which has (positive) solution

$$x = -y^2 + \sqrt{y^4-y+1}.$$

In other words,

$$\int_0^1 \frac{\sqrt{1+8x-8x^3}}{4x} - \frac{1}{4x}\,dx = \int_0^1 -y^2 + \sqrt{y^4-y+1}\,dy.$$

Hence the given integral equals

$$\int_0^1 -y^2\,dy = -\frac{1}{3} \in \mathbb{Q}.$$

Given a depressed quartic $y^4 + a_2y^2 + a_1y + a_0$, we express it as

$$y^4 + a_2y^2 + a_1y + a_0 = \Delta_x \left(\frac{1}{4}x^2 + xy^2 - (a_2y^2 + a_1y + a_0) \right)$$

where Δ_x denotes the discriminant as a polynomial in x. If we compute its discriminant as a polynomial in y, we have

$$\Delta_y \left((x - a_2)y^2 - a_1y + (\frac{1}{4}x^2 - a_0) \right) = -x^3 + a_2x^2 + 4a_0x + a_1^2$$

which is exactly the negative of the resolvent cubic of the original quartic!

14: (4 minutes)

A positive integer is a Gian's integer if it is of the form $a^4 + b^3$ for some positive integers a, b.

T/F: For any integer $n \geq 3$, there exist infinitely many integers m such that there are exactly $n+1$ Gian's integers among $m+1, m+2, \ldots, m+n^3$.

14: (4 minutes)

A positive integer is a Gian's integer if it is of the form $a^4 + b^3$ for some positive integers a, b.

T/F: For any integer $n \geq 3$, there exist infinitely many integers m such that there are exactly $n+1$ Gian's integers among $m+1, m+2, \ldots, m+n^3$.

T: For each $m \in \mathbb{N}$, let $G(m)$ denote the number of Gian's integers among $m+1, m+2, \ldots, m+n^3$. Note that $G(m+1) - G(m) \in \{-1, 0, 1\}$ for all $m \in \mathbb{N}$. The key idea is to construct two increasing sequences $(s_i)_{i=1}^{\infty}$ and $(t_i)_{i=1}^{\infty}$ such that $G(s_i) = 0$ and $G(t_i) \geq n+1$ for all i. Suppose we have such two sequences. Then for any s_i, we can find some $t_j > s_i$ and then there exists $m \in [s_i, t_j]$ such that $G(m) = n+1$.

We construct the s_i first. Let $N \geq 2$ be any positive integer. Then there are less than N^7 Gian's integers in $[0, N^{12}]$. Divide the interval $[0, N^{12}]$ into N^7 intervals of length N^5. Then at least one of them contains no Gian's integers and it's not $[0, N^5]$. In other words, if $N^5 > n^3$, then there exists some $m \in [N^5, N^{12} - N^5]$ such that $G(m) = 0$. Take $N_1 = n$ and choose N_{i+1} so that $N_{i+1}^5 > N_i^{12} - N_i^5$. Then we may find s_i in the disjoint intervals $[N_i^5, N_i^{12} - N_i^5]$.

To construct the t_i, we simply take $t_i = i^{12}$. Then $i^{12} + 1^3, \ldots, i^{12} + n^3, 2^4 + i^{12}$ are all $n+1$ distinct Gian's integers as $2^4 < n^3$ and 2^4 is not a cube.

15: (5 minutes)

For any positive integer m, let $S(m)$ be the number of positive integers $n < \operatorname{lcm}(1, 2, \ldots, m)$ such that its remainders when divided by $2, 3, \ldots, m$ are all distinct.

T/F: $S(2025) - 1$ is a power of 2.

15: (5 minutes)

For any positive integer m, let $S(m)$ be the number of positive integers $n < \text{lcm}(1, 2, \ldots, m)$ such that its remainders when divided by $2, 3, \ldots, m$ are all distinct.

T/F: $S(2025) - 1$ is a power of 2.

T: The integer n is determined by its remainders mod $2, 3, \ldots, m$. So we are really counting the number of possible remainders.

Claim: If $p \mid n$ for some $p = 3, \ldots, m$, then $p > m/2$ is a prime.
Proof: Note that n is also divisible by any divisor of p. So p must be prime. Moreover for $k = 2, 3, \ldots, p-1$, we must have

$$n \equiv k - 1 \equiv -1 \pmod{k}.$$

Now $p + 1$ is not a prime and for any proper divisor $d \mid p + 1$, we have $d < p$ and so $n \equiv -1 \pmod{d}$. In other words,

$$n \equiv -1 \equiv p \pmod{p+1}.$$

Note that we have used 0 and p. Since $p \mid n$, there is no more possible remainder for n mod $2p$. Therefore, $2p > m$. $\square$

Consider first the number of odd n. Now $n \equiv 1 \pmod{2}$. The Claim above implies that the remainder 0 cannot appear mod k for any $k \leq m/2$. So we must have $n \equiv k - 1 \equiv -1 \pmod{k}$ for all such k. For any composite $k \in (m/2, m]$ with at least two prime divisors, all of its proper divisors are at most $m/2$ and so $n \equiv -1 \pmod{k}$. Suppose next $k = p^s \in (m/2, m]$ for some prime p and $s \geq 2$. We know $p < m/2$ and so $n \equiv -1 \pmod{p}$, implying that $n \equiv \ell p - 1 \pmod{p^2}$ for some $\ell = 2, \ldots, p$. However for $\ell = 2, \ldots, p - 1$, we have $n \equiv -1 \pmod{\ell p}$ since they have at least two prime divisors. Hence $n \equiv -1 \pmod{p^2}$. The same argument then can be repeated to give $n \equiv -1 \pmod{p^s}$.

Let $p_1 < \cdots < p_t$ be all the primes in $(m/2, m]$. The possible remainders left are $\{0, p_1 - 1, \ldots, p_t - 1\}$. There are two choices (0 or $p_1 - 1$) for the remainder mod p_1, and once that's chosen, there are two choices for p_2 and so on. Hence, we have 2^t possible odd n's.

Consider now the number of even n. Since $2 \mid n$, the remainders of n mod all even numbers are also even. Hence it is easy to see that $n \equiv k - 2 \pmod{k}$ for all even k and then also $n \equiv k - 2 \pmod{k}$ for all odd k. Hence there is only one possible even n brining the total number to

$$2^{\pi(m) - \pi(m/2)} + 1.$$

16: Pizza/Tie break, if needed (3 minutes)

Compute

$$\sum_{k=1}^{8} e^{-k^2\pi/9}.$$

16: Pizza/Tie break 2, if needed (3 minutes)

Compute

$$\sum_{k=1}^{2024} e^{-k^2\pi/2025}.$$

16: Pizza/Tie break, if needed (3 minutes)

Compute

$$\sum_{k=1}^{8} e^{-k^2\pi/9}.$$

```python
# Online Python compiler (interpreter) to run Python online.
# Write Python 3 code in this online editor and run it.
import math

n = 9
sum_S = 0

for k in range(1, n):  # Range is from 1 to 2024 (inclusive)
    term = math.exp(-k*k*math.pi / n)
    sum_S += term

print(sum_S)

```

```
1.0000000000010505

=== Code Execution Successful ===
```

It can be shown that

$$\sum_{k=1}^{n-1} e^{-k^2\pi/n} \approx \frac{-1+\sqrt{n}}{2}.$$