Bernoulli Trial 2025

1: (2 minutes) **T/F**: There are infinitely many positive integers d, n such that $d \mid n$ and

$$\binom{2d}{d} \nmid \binom{2n}{n}.$$

2: (3 minutes) **T/F**: $x^{46} + 69x + 2025$ is irreducible in $\mathbb{Z}[x]$.

3: (3 minutes) **T/F**: There is a unique digit $d = 1, \ldots, 9$ such that if 2^n and 5^n start with the same digit for some $n \in \mathbb{N}$, then that digit is d.

4: (4 minutes) **T/F**: For any odd prime number p,

$$\left(\frac{p-1}{2}\right)^3 \mid \sum_{o=1}^{p-1} \sum_{r=1}^{p-1} \sum_{z=1}^{p-1} \left\lfloor \frac{orz}{p} \right\rfloor.$$

5: (4 minutes) **T/F**: $\displaystyle\int_2^\infty \frac{1}{x^7 - x}\, dx > \frac{1}{365}$.

6: (4 minutes) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables uniform on $(0, 1)$. Let

$$R_n = \sum_{k=1}^{n} \begin{cases} 1 & \text{if } X_k = \max\{X_1, \ldots, X_k\} \\ 0 & \text{otherwise} \end{cases}$$

For any random variable Y, let $E(Y)$ denote its expectation and $\mathrm{Var}(Y) = E(Y^2) - E(Y)^2$ denots its variance.

T/F: $\displaystyle\lim_{n \to \infty} \Big(E(R_n) - \mathrm{Var}(R_n) \Big) < \frac{\pi^2}{420/69}$.

7: (5 minutes) **T/F**: For any integers $n, d \geq 3$, there exists a set $S \subseteq \{1, 2, \ldots, (2n)^d\}$ of size at least n^{d-2}/d that does not contain any 3-term arithmetic progression (i.e. there does not exist $a, b, c \in S$ such that $a + b = 2c$).

8: (2 minutes) Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ be four distinct points on (both branches of) the hyperbola $xy = 1$. Suppose they lie on a circle.

T/F: $x_1 x_2 x_3 x_4 = 1$.

9: (3 minutes) **T/F**: There exist **unique** bijections $f, g, h : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$f(n)^3 + g(n)^3 + h(n)^3 = 3ng(n)h(n).$$

10: (4 minutes) For any positive integer n, let S_n denote the group of all permutations of $\{1,\ldots,n\}$. For each $\sigma \in S_n$, let $\mathrm{Orb}(\sigma)$ denote the number of cycles of σ (which is the same as the number of orbits as σ acts on $\{1,\ldots,n\}$).

T/F: $$\frac{1}{69!}\sum_{\sigma\in S_{69}} \mathrm{Orb}(\sigma) < 4.$$

11: (3 minutes) **T/F**: There does not exist $B \in M_{69\times 69}(\mathbb{R})$ such that $\dim_{\mathbb{R}}(\{BAB\colon A \in M_{69\times 69}(\mathbb{R})\}) = 2025$.

12: (3 minutes) **T/F**: There exists $a, b, c \in \mathbb{Z}$ such that $|\zeta_{13}^a + \zeta_{13}^b + \zeta_{13}^c + 1| = \sqrt{3}$, where $\zeta_{13} = e^{2\pi i/13}$.

13: (4 minutes) **T/F**: $\displaystyle\int_0^1 \frac{\sqrt{1+8x-8x^3}}{4x} - \sqrt{x^4-x+1} - \frac{1}{4x}\,dx \notin \mathbb{Q}$.

14: (4 minutes) A positive integer is a Gian's integer if it is of the form a^4+b^3 for some positive integers a, b.

T/F: For any integer $n \geq 3$, there exist infinitely many integers m such that there are exactly $n+1$ Gian's integers among $m+1, m+2, \ldots, m+n^3$.

15: (5 minutes) For any positive integer m, let $S(m)$ be the number of positive integers $n < \mathrm{lcm}(1,2,\ldots,m)$ such that its remainders when divided by $2, 3, \ldots, m$ are all distinct.

T/F: $S(2025) - 1$ is a power of 2.

16: Tie break, if needed (3 minutes) Compute $\displaystyle\sum_{k=1}^{8} e^{-k^2\pi/9}$.