Bernoulli Trial 2023

1: (2 minutes)

T/F: There is a complex number z with $|z| = 1$ such that

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F: Since $1, z^{2023}, z^4$ all have length 1, they must form an equilateral triangle if they add up to 0. So $\{z^{2023}, z^4\} = \{\zeta_3, \zeta_3^2\}$. Then $z = z^{2024-2023} \in \{1, \zeta_3, \zeta_3^2\}$ but then since $2023 \equiv 4 \pmod 3$, we have $z^{2023} = z^4$. Contradiction.

Using the same argument, one can show that if $\gcd(n, m) = 1$, then $z^m + z^n + 1 = 0$ has a solution with $|z| = 1$ if and only if $n \equiv 1, m \equiv 2 \pmod 3$ or $n \equiv 2, m \equiv 1 \pmod 3$.

2: (2 minutes)

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T: Let $s(n)$ denote the sum of the digits of n. Since $\log_{10}(2023) < 3.5$ as $\sqrt{10} > 3$, we see that 2023^{2023} has less than 7100 digits, which means that $s(2023^{2023}) < 7100 \times 9 = 63900$. Then $s(s(2023^{2023})) < s(69999) = 42$ and $s(s(s(2023^{2023}))) < s(49) = 13$. Next we compute $2023^{2023} \bmod 9$. Note that $2023 \equiv 7 \pmod 9$ and $7^3 \equiv 1 \pmod 9$ and so $2023^{2023} \equiv 7^1 = 7 \pmod 9$. The only positive integer less than 13 that is congruent to 7 modulo 9 is 7.

This question is from IMO 1975 P4 where the number was 4444^{4444}; the exact same analysis applies.

3: (3 minutes)

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T/F: There is a unique 3-digit cubical number n such that $n+1$ is also cubical.

T: There are three possibilities:

$$\begin{aligned} a^3 + b^3 + (c+1)^3 &= a^3 + b^3 + c^3 + 1, \\ a^3 + (b+1)^3 + 0^3 &= a^3 + b^3 + 9^3 + 1, \\ (a+1)^3 + 0^3 + 0^3 &= a^3 + 9^3 + 9^3 + 1. \end{aligned}$$

The last two cases are not possible because the difference of two consecutive cubes of single digit numbers is too small to cover the loss of 9^3. The first possibility gives $c = 0$. So now we have $100a + 10b = a^3 + b^3$. Checking some small values gives $3^3 + 7^3 = 370$.

It turns out that $10 \mid a^3 + b^3$ if and only if $10 \mid a + b$. The only cubical numbers are $1, 153, 370, 371, 407$.

4: (3 minutes)

T/F:

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T: Consider the substitution $u = 1/x$. We have

$$\int_0^\infty \frac{\ln(2x)}{1+x^2}\,dx = \int_0^\infty \frac{\ln(2u^{-1})}{1+u^2}\,du.$$

Their sum is

$$2\ln 2 \int_0^\infty \frac{1}{1+x^2}\,dx = \pi \ln 2 < \pi.$$

5: (3 minutes)

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F: The number i cannot be written in this form. First it is easy to see that if a number can be written as a sum of distinct powers of $1 + i$, such a representation must be unique, because $1 + i$ is not a unit. Next we observe that $i - 1 = i(1 + i)$. This means that if $i = a_0 + a_1(1 + i) + \cdots + a_n(1 + i)^n$ with $a_n \neq 0$, then

$$a_0 + a_1(1 + i) + \cdots + a_n(1 + i)^n = 1 + a_0(1 + i) + a_1(1 + i)^2 + \cdots + a_n(1 + i)^{n+1}.$$

Hence $a_n = 0$. Contradiction.

It turns out that exactly one number out of z and $i - z$ can be written as a finite sum of distinct powers of $1 + i$. To prove this, consider the function

$$f(a + bi) = \begin{cases} (a + bi)/(1 + i) & \text{if } a \equiv b \pmod 2; \\ (a - 1 + bi)/(1 + i) & \text{if } a \not\equiv b \pmod 2. \end{cases}$$

Then it is easy to see that z can be written if and only if $f(z)$ can be written. It is also not hard to prove that the sequence $z_1 = f(z)$, $z_{n+1} = f(z_n)$ is eventually constant, and equals to 0 or i. Finally, we have $f(i - z) = i - f(z)$.

6: (3 minutes)

Let n be a positive integer such that $n \equiv 6 \pmod 7$.

T/F: The equation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

has solutions in $x, y, z \in \mathbb{N}$.

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T: Write $n + 1 = 7k$ for some $k \in \mathbb{N}$. Dividing by kn gives $\frac{1}{k} + \frac{1}{kn} = \frac{7}{n}$. So $\frac{1}{n} + \frac{1}{k} + \frac{1}{kn} = \frac{8}{n}$. Dividing by 2 gives

$$\frac{1}{2n} + \frac{1}{2k} + \frac{1}{2kn} = \frac{4}{n}.$$

Erdös-Straus conjectured that this equation is solvable for all positive integers $n \geq 2$. This conjecture is currently open. For $X > 0$, let $N(X)$ be the number of positive integers $n < X$ such that this equation is not solvable. Then by considering more primes, one can prove

$$N(X) \ll_\epsilon \frac{X}{(\log X)^{9/4-\epsilon}}.$$

A useful related result is that the equation $a/b = 1/x + 1/y$ is solvable if there are divisors d_1, d_2 of b such that $a \mid d_1 + d_2$. Indeed, let $k = (d_1 + d_2)/a$ and take $x = kb/d_1$ and $y = kb/d_2$.

7: (5 minutes)

T/F: There exists a set

$$A \subseteq \{(i,j) \in \mathbb{Z}^2 \colon 1 \le i \le 2023, 1 \le j \le 2023\}$$

such that for any $i, j = 1, \ldots, 2023$, there exist exactly 7 integers k such that $(i,k) \in A$ and $(k,j) \in A$.

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T: Let $n = 2023$ and $m = 7$. Since $2023 = 7 \times 17^2$, it might help to consider the easier case where $n = 17^2$ and $m = 1$. Let M_A be the $n \times n$ matrix whose (i,j) entry is 1 if $(i,j) \in A$ and 0 if otherwise. For any positive integer d, let J_d denote the $d \times d$ matrix with 1's everywhere. We are then looking for a set A such that $M_A^2 = mJ_n$.

We claim first that if such an A exists for (n,m), then it also exists for (dn, dm) for any positive integer d. Indeed, simply take the $dn \times dn$ matrix $M_{A'}$ such that all of its $n \times n$ blocks are M_A. Then it is easy to see that $M_{A'}^2 = dmJ_{dn}$. Moreover, the entries of $M_{A'}$ are all 1 and 0, so it comes from a set A'.

It now remains to construct the set A when $n = 17^2$ and $m = 1$. We observe that every integer between 1 and 17^2 can be written uniquely as $17(q-1) + r$ for some $q, r = 1, \ldots, 17$. We let

$$A = \{(17(q-1) + r, 17(r-1) + d) \colon 1 \leq q, r, d \leq 17\}.$$

Then given $i = 17(q_1 - 1) + r_1$ and $j = 17(q_2 - 1) + r_2$, the unique integer k such that $(i, k), (k, j) \in A$ is $k = 17(r_1 - 1) + q_2$.

For which other pairs (n, m) is this possible? The matrix mJ_n should have an integral square root. The eigenvalues of mJ_n are $0, \ldots, 0, mn$. So we need mn to be a square since the trace of M_A is an integer. Since the trace of M_A is at most n, we also have $m \leq n$. In other words, we need $n = dt^2$ and $m = ds^2$ for some coprime integers s, t with $s \leq t$. We have already seen that the extra common factor of d is harmless. Let's consider $n = t^2$ and $m = s^2$. We use the same idea and write every integer from 1 to t^2 uniquely as $t(q-1) + r$ for some $q, r = 1, \ldots, t$. Then given $i = t(q_1 - 1) + r_1$ and $j = t(q_2 - 1) + r_2$, to find s^2 integers k such that $(i, k), (k, j) \in A$, we ideally want $k = t(q_3 - 1) + r_3$ where there are s choices for q_3 and s choices for r_3. To arrange for this, we let B be any subset of $\{1, \ldots, t\}$ of size s. Then we take

$$A = \{(t(q-1) + r, t(q'-1) + r') \colon 1 \leq q, r, q', r' \leq t, \quad q' - r \equiv b \pmod{t} \text{ for some } b \in B\}.$$

Therefore, such a set A exists if and only if mn is a square and $m \leq n$.

8: (2 minutes)

T/F: There exists a polynomial $f(x) \in \mathbb{Z}[x]$, an integer $n \geq 3$, and distinct integers $a_1, \ldots, a_n$ such that

$$f(a_i) = a_{i+1} \text{ for } i = 1, \ldots, n-1$$

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F: Standard polynomial division result tells us that

$$a_2 - a_1 \mid a_3 - a_2 \mid \cdots \mid a_n - a_{n-1} \mid a_1 - a_n \mid a_2 - a_1.$$

Hence there is a nonzero integer c such that all the above differences equal $\pm c$. Since they add up to 0, they can't all be c or $-c$. So there exists an index $i \pmod{n}$ such that $a_i - a_{i-1} = -(a_{i+1} - a_i)$, which then implies $a_{i+1} = a_{i-1}$, contradicting the assumption that they are distinct.

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T: Suppose $n, k \in \mathbb{N}$ and

$$d = \gcd(2^{2^n}+1, 2^{2^{n+k}}+1).$$

Then $d \mid 2^{2^{n+1}}-1$ and so $d \mid 2^{2^{n+k}}-1$. This implies $d \mid 2$. So $d=1$.

Note that if p is a prime divisor of $2^{2^n}+1$, then $o_p(2) \mid 2^{n+1}$ in $(\mathbb{Z}/p\mathbb{Z})^\times$. Since $p \nmid 2^{2^n}-1$, we see that $o_p(2) = 2^{n+1}$. So $2^{n+1} \mid p-1$. In fact, we can also prove that $2^{n+2} \mid p-1$. Since $p \equiv 1 \pmod 8$, we know that 2 is a square mod p. Let $a \in \mathbb{F}_p$ such that $a^2 = 2$. Then $a^{2^{n+2}} = 2^{n+1} = 1$ in $\mathbb{Z}/p\mathbb{Z}$ and $a^{2^{n+1}} = 2^n \neq 1$. So $o(a) = 2^{n+2}$ and it divides $p-1$.

10: (4 minutes)

T/F:

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F: The integrand has a maximum of 2^{n-1} at $1/2$ and decreases to 1 when $x = 0$ and $x = 1$. This suggests setting $u = x - 1/2$ and then $v = 2u$ to get

$$\frac{n}{2^n} \int_0^1 \frac{dx}{x^n + (1-x)^n} = \frac{n}{2} \int_{-1}^1 \frac{dv}{(1+v)^n + (1-v)^n}.$$

Next we set $w = nv$ to get

$$\frac{1}{2} \int_{-n}^{n} \frac{dw}{(1+w/n)^n + (1-w/n)^n}$$

which we expect will converge to

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{dw}{e^w + e^{-w}} = \frac{1}{2} \arctan(e^w)\Big|_{-\infty}^{\infty} = \frac{\pi}{4}.$$

To make the convergence rigorous, we use Lebesgue Dominated Convergence Theorem. Let

$$f_n(w) = \frac{\chi_{[-n,n]}(w)}{(1+w/n)^n + (1-w/n)^n}.$$

Then

$$f_n(w) \leq \frac{1}{(1+|w|/n)^n} \leq \frac{1}{(1+|w|/2)^2}$$

for $n \geq 2$ and

$$\int_{-\infty}^{\infty} \frac{dw}{(1+|w|/2)^2} < \infty.$$

So

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(w)\, dw = \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(w)\, dw = \int_{-\infty}^{\infty} \frac{dw}{e^w + e^{-w}}.$$

11: (4 minutes)

Let $A \subseteq \mathbb{Z}^2$ be a set such that any open disc of radius 2023 contains at least one point in A.

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T/F: For any coloring of the points in A with 11 colors, there exist 4 points in A with the same color and they form a rectangle.

T: Consider a huge square with side length $4046L$ with sides parallel to the coordinate axes. We can divide it into L^2 squares of side length 4046 and fit a disc of radius 2023 inside each of it. Hence, this square contains at least L^2 points in A. There are $4046L + 1$ vertical grid lines in this square. So there exists a vertical grid line with at least $L^2/(4046L + 1)$ points in A. By taking L large enough, say $L = 50000$, there is a vertical grid line inside the box with at least 12 points in A, so then at least 2 points in A with the same color. There are only finitely many possible configurations for 2 lattice points on a vertical line of length $4046 \cdot 50000$ having one of the 11 colors, but there are infinitely many non-overlapping squares with side length $4046 \cdot 50000$ that we can line up horizontally.

Obviously the numbers 2023 and 11 don't matter.

12: (4 minutes)

A fair die (so that it has $1/6$ chance of rolling each $1, 2, 3, 4, 5, 6$) is rolled infinitely. For any positive integer n, let a_n be the probability that a partial sum of n is reached.

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F: We have the recursion formula

$$a_{n+6} = \frac{1}{6}a_n + \frac{1}{6}a_{n+1} + \cdots + \frac{1}{6}a_{n+5}$$

where we put $a_0 = 1$ and $a_n = 0$ for $n < 0$. Its generating function is then given by

$$F(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{6}{6 - x - x^2 - \cdots - x^6}.$$

We observe that

$$6 - x - x^2 - \cdots - x^6 = (1-x)(6 + 5x + 4x^2 + 3x^3 + 2x^4 + x^5) = (1-x)(x - r_1)\cdots(x - r_5)$$

where $|r_1|, \ldots, |r_5| > 1$. Applying partial fraction decomposition gives

$$\frac{6}{6 - x - x^2 - \cdots - x^6} = \frac{A}{1-x} + \sum_{i=1}^{5} \frac{B_i}{x - r_i} = \frac{A}{1-x} - \sum_{i=1}^{5} \frac{B_i/r_i}{1 - x/r_i}$$

for some constants $A, B_1, \ldots, B_5$. Multiplying by $6 - x - x^2 - \cdots - x^6$ and setting $x = 1$ gives $A = 2/7$. Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{2}{7} - \sum_{i=1}^{5} \frac{B_i}{r_i^{n+1}} \right) = \frac{2}{7}.$$

Finally

$$\frac{2}{7} = \frac{1}{11}\frac{22}{7} = \frac{3.142857\ldots}{11} > \frac{\pi}{11}.$$

Note that by working with the recursion formula, one can also show that the limit, if exists, must equal $2/7$, which is enough to conclude that the given statement is false.

13: (4 minutes)

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$$\sum_{n=0}^{17} n^{2023}\binom{17}{n}(-1)^n \text{ is divisible by } 17!.$$

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T: Note that

$$\frac{1}{17!}\binom{17}{n}(-1)^n = \frac{(-1)^n}{n!(17-n)!} = \prod_{\substack{m \neq n \\ 0 \leq m \leq 17}} \frac{1}{m-n}.$$

Consider now

$$f(x) = \sum_{n=0}^{17} n^{2023} \prod_{\substack{m \neq n \\ 0 \leq m \leq 17}} \frac{m-x}{m-n}.$$

Then $f(x)$ is a polynomial of degree at most 17 with $f(n) = n^{2023}$ for $n = 0, \ldots, 17$. Our goal is to show that its x^{17}-coefficient is an integer. In fact, we prove that $f(x) \in \mathbb{Z}[x]$. Applying the division algorithm to x^{2023} by $(x-0)\cdots(x-17)$ gives $q(x), r(x) \in \mathbb{Z}[x]$ with $\deg r \leq 17$ and

$$x^{2023} = (x-0)\cdots(x-17)q(x) + r(x).$$

Then $r(n) = n^{2023}$ for $n = 0, \ldots, 17$. So $r(x) = f(x)$.

14: (4 minutes)

T/F: For any continuous function $g(x) : [-1, 1] \to \mathbb{R}$,

$$\left(\int_{-1}^{1} g(x)\, dx \right)^2 + \left(\int_{-1}^{1} x g(x)\, dx \right)^2 \leq 2 \int_{-1}^{1} g(x)^2\, dx.$$

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T: Note that without the second term on the LHS, this is just Cauchy-Schwartz. So perhaps we should use the more complete version. There is an orthonormal sequence $\{P_n(x)\}_{n=0}^{\infty}$ of polynomials such that $\deg(P_n(x)) = n$ and

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \delta_{nm}.$$

More precisely, we have $P_0(x) = \dfrac{1}{\sqrt{2}}$, $P_1(x) = \dfrac{\sqrt{3}}{\sqrt{2}}x$.

Since continuous functions can be approximated by polynomials (in L^∞), it is enough to consider polynomials $g(x)$, in which case we can write $g(x) = a_0P_0(x) + \cdots + a_dP_d(x)$ where $d = \deg(g(x))$. Now the desired inequality is

$$2a_0^2 + \frac{2}{3}a_1^2 \le 2(a_0^2 + a_1^2 + \cdots + a_d^2)$$

which is clearly true.

15: (5 minutes)

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T: Let $P(x)$ denote the largest prime divisor of x. The key starting point is the factorization

$$(2m^2)^2 + 1 = (2m^2 - 2m + 1)(2m^2 + 2m + 1).$$

So when n is of the form $2m^2$, $P(n^2 + 1)$ is already at most around n. To lower it further, we want to find m so that $2m^2 - 2m + 1$ and $2m^2 + 2m + 1$ have large prime divisors.

Lemma: Let $f(x) \in \mathbb{Z}[x]$ be a non-constant polynomial. Then there are infinitely many primes p dividing $f(a)$ for some $a \in \mathbb{Z}$.

Proof: Let $a_0 = f(0)$. If $a_0 = 0$, then $p \mid f(p)$ for all primes p. Suppose $a_0 \neq 0$. Then $f(a_0 n!) = a_0(1 + n! g(n!))$ has a prime divisor $p > n$ for n large enough.

Let ℓ be big enough so that $p_1 = P(2\ell^2 - 2\ell + 1) > 2023/\epsilon$. Then $p_1 \mid 2(\ell + tp_1)^2 - 2(\ell + tp_1) + 1$ for any $t \in \mathbb{Z}$. We can not take t large enough so that for $k = \ell + tp_1$, $q_1 = P(2k^2 - 2k + 1) \geq p_1 > 2023/\epsilon$ and

$q_2 = P(2k^2+2k+1) > 2023/\epsilon$. The same is also for any $m = k + sq_1q_2$. Now $q_2 \mid 2m^2+2m+1$ and so

$$P(2m^2+2m+1) \le \max\{q_2, \frac{2m^2+2m+1}{q_2}\} < \epsilon(2m^2).$$

Similarly for $P(2m^2-2m+1)$.

The more interesting question is of course the conjecture that n^2+1 is prime infinitely often. The best result currently (2020) is that $P(n^2+1) > n^{1.279}$ for infinitely many n.