Bernoulli Trial 2023

1: (2 minutes) **T/F**: There is a complex number z with $|z| = 1$ such that $z^{2023} + z^4 + 1 = 0$.

2: (2 minutes) **T/F**: The sum of the digits of the sum of the digits of the sum of the digits of 2023^{2023} is 7.

3: (3 minutes) A *cubical* number is a positive integer that is equal to the sum of the cubes of its digits.

T/F: There is a unique 3-digit cubical number n such that $n+1$ is also cubical.

4: (3 minutes) **T/F**:

$$\int_0^\infty \frac{\ln(2x)}{1+x^2}\,dx < \frac{\pi}{2}.$$

5: (3 minutes) **T/F**: Every Gaussian integer $a + bi$ with $a, b \in \mathbb{Z}$ can be written as a finite sum of distinct powers of $1 + i$.

6: (3 minutes) Let n be a positive integer such that $n \equiv 6 \pmod 7$.

T/F: The equation $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ has solutions in $x, y, z \in \mathbb{N}$.

7: (5 minutes) **T/F**: There exists a set

$$A \subseteq \{(i,j) \in \mathbb{Z}^2 \colon 1 \le i \le 2023, 1 \le j \le 2023\}$$

such that for any $i, j = 1, \ldots, 2023$, there exist exactly 7 integers k such that $(i,k) \in A$ and $(k,j) \in A$.

8: (2 minutes) **T/F**: There exists a polynomial $f(x) \in \mathbb{Z}[x]$, an integer $n \ge 3$, and distinct integers $a_1, \ldots, a_n$ such that $f(a_i) = a_{i+1}$ for $i = 1, \ldots, n-1$ and $f(a_n) = a_1$.

9: (2 minutes) A *Fermat number* is a number of the form $2^{2^n} + 1$ for some non-negative integer n.

T/F: Every two distinct Fermat numbers are coprime.

10: (4 minutes) **T/F**:

$$\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{dx}{x^n + (1-x)^n} < \frac{\pi}{4}.$$

11: (4 minutes) Let $A \subseteq \mathbb{Z}^2$ be a set such that any open disc of radius 2023 contains at least one point in A.

T/F: For any coloring of the points in A with 11 colors, there exist 4 points in A with the same color and they form a rectangle.

12: (4 minutes) A fair die (so that it has $1/6$ chance of rolling each $1, 2, 3, 4, 5, 6$) is rolled infinitely. For any positive integer n, let a_n be the probability that a partial sum of n is reached.

T/F:

$$\lim_{n\to\infty} a_n < \frac{\pi}{11}.$$

13: (4 minutes) **T/F**:

$$\sum_{n=0}^{17} n^{2023}\binom{17}{n}(-1)^n \text{ is divisible by } 17!.$$

14: (4 minutes) **T/F**: For any continuous function $g(x) : [-1, 1] \to \mathbb{R}$,

$$\left(\int_{-1}^{1} g(x)\,dx\right)^2 + \left(\int_{-1}^{1} xg(x)\,dx\right)^2 \le 2\int_{-1}^{1} g(x)^2\,dx.$$

15: (5 minutes) **T/F**: For any $\epsilon > 0$, there are infinitely many positive integers n such that the largest prime factor of $n^2 + 1$ is at most ϵn.