Bernoulli Trial 2022

1: (2 minutes)

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T: Suppose for a contradiction that $x \in B \backslash A$. Then $x \in B \cup D$ and so $x \in C$. Moreover, $x \notin A \cap C$, so $x \notin B \cap D$ and so $x \notin D$. This contradicts $C \subseteq D$.

If we only have $A \cup C = B \cup D$ or $A \cap C = B \cap D$, then we do not have the same result.

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F: Note that the sum of the digits of a number is congruent to the same number mod 3. If $s(n) = s(n+1)$, then $2^{n+1} \equiv 2^n \pmod 3$, which implies $2 \equiv 1 \pmod 3$, which is a contradiction.

If we work mod 9, we find $s(n) = s(m)$ implies that $n \equiv m \pmod 6$. In fact, $2^{12} = 4096$ and $2^{18} = 262144$ have the same sum of digits. Are there infinitely such pairs?

3: (3 minutes)

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T: A composite odd positive integer is of the form $(2r+1)(2r+2s+1)$. If we fix r and let s vary, then we obtain an arithmetic progression with initial term $(2r+1)^2$ and common difference $2(2r+1)$. We then obtain three arithmetic progressions with $r = 1, 2, 3$. Their union contains every composite odd positive integer that is a multiple of 3 or 5 or 7, which is true for every composite odd positive integer less than 121.

Let p_n denote the n-th odd prime. Is it true that $N = p_{n+1}^2 + 2$ is the smallest positive integer such that the set of all composite odd positive integers less than N cannot be written as a union of n arithmetic progressions?

4: (4 minutes)

If $x < y < z$ are positive integers such that $4^x + 4^y + 4^z$ is a square, then $z - 2y + x = -1$.

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T: WLOG, we may assume $x = 0$. So $1 + 4^y + 4^z = (1 + 2m)^2$ for some integer m, which gives $4^{y-1} + 4^{z-1} = m + m^2$. It now suffices to consider $4^a + 4^b = m(m+1)$ with $0 \leq a < b$ and we aim to prove $b = 2a$. We first prove that $b \geq 2a$. If $a = 0$, then it is automatically true. Suppose $a > 0$. Since $m(m+1) = (-m-1)(-m)$, we may assume $4^a \mid m$. So $m = 4^a c$ for some integer c. Now $1 + 4^{b-a} = c(1 + 4^a c)$ and so $c = 1 + 4d$ for some integer d. Then $4^{b-a} = 4d + (1 + 4d)^2 4^a = 4^a + (8d + 16d^2)4^a + 4d \geq 4^a$. So $b \geq 2a$.

Suppose $b > 2a$ for a contradiction. Then if $m \geq 2^b$, we have $m(m+1) \geq 4^b + 2^b > 4^b + 4^a$ and if $0 \leq m \leq 2^b - 1$, we have $m(m+1) \leq 4^b - 2^b < 4^b + 4^a$. Therefore, $4^b + 4^a$ cannot be written as $m(m+1)$.

For which positive integer n does $n^x + n^y = z(z+1)$ have nontrivial positive integer solutions? Here a trivial solution is a solution where $y = 2x$ and $z = n^x$ or $x = 2y$ and $z = n^y$. The same argument shows that when n is a square and has a unique prime divisor, there are no nontrivial solutions. When $n = 2$, we have $4 + 8 = 3(4)$ and $4 + 128 = 11(12)$. Are there infinitely many solutions when $n = 2$? What about $n = 3$? $(3 + 27 = 5(6).)$

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T: Using the binary expansion of n, we may write $n = 2^{a_0} + 2^{a_1} + \cdots + 2^{a_k}$ with $0 \le a_0 < a_1 < \cdots < a_k$. Then working mod 2, we have

$$(1+x)^n = \prod_{i=0}^{k}(1+x)^{2^{a_i}} = \prod_{i=0}^{k}(1+x^{2^{a_i}}) = \sum_{S \subset \{0,\ldots,k\}} x^{\sum_{i\in S} 2^{a_i}}.$$

From the uniqueness of binary representation, we see that the exponent $\sum_{i\in S} 2^{a_i}$ for different S's are all distinct. Hence, there are 2^{k+1} nonzero terms in $(1+x)^n$ mod 2. These are exactly the odd binomial coefficients.

Any pattern for the number of binomial coefficients that are divisible by 3?

6: (3 minutes)

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F: Suppose for a contradiction that the statement is true. Let N be a large enough integer satisfying the above condition. Let $p_1 < p_2 < \cdots < p_n$ be all the primes less than or equal to $\sqrt{N}$. Since $p_i^2 \leq N$ and p_i^2 is not a prime, we have $\gcd(p_i^2, N) \neq 1$ and so $p_i \mid N$. Hence $p_1 p_2 \cdots p_n \mid N$. We now have a contradiction because the product of the first n primes grows way faster than the square of the n-th prime.

We can try lowering the bound for k. For example, if $f(x)$ is a function such that $f(n!) < n$, then by taking $N = n!$, then we see that any $k \leq f(N)$ is less than n is so $\gcd(k, n!) \neq 1$. How big can we take $f(x)$ so that there are infinitely many positive integers N such that $1 < k \leq f(N)$ and $\gcd(k, N) = 1$ implies k is a prime.

7: (3 minutes)

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F:

$$\left(\sum_{r=1}^{\infty} \frac{1}{r^2}\right)^2 = \sum_{m,n=1}^{\infty} \frac{1}{m^2n^2} = \sum_{d=1}^{\infty} \sum_{\substack{m,n=1\\ \gcd(m,n)=1}}^{\infty} \frac{1}{(dmdn)^2} = \sum_{d=1}^{\infty} \frac{1}{d^4} \sum_{\substack{m,n=1\\ \gcd(m,n)=1}}^{\infty} \frac{1}{(mn)^2}$$

Hence

$$\sum_{\substack{m,n=1\\ \gcd(m,n)=1}}^{\infty} \frac{1}{(mn)^2} = \left(\frac{\pi^2}{6}\right)^2 \left(\frac{\pi^4}{90}\right)^{-1} \in \mathbb{Q}.$$

Is there a more intuitive reason for this?

8: (3 minutes)

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T: Let α, β be two irrational numbers such that $\{1, \alpha, \beta\}$ is linearly independent over $\mathbb{Q}$. We claim that any two distinct lattice points (x_1, y_1) and (x_2, y_2) must have distinct distance to (α, β). Suppose

$$(x_1 - \alpha)^2 + (y_1 - \beta)^2 = (x_2 - \alpha)^2 + (y_2 - \beta)^2.$$

Then

$$2(x_1 - x_2)\alpha + 2(y_1 - y_2)\beta \in \mathbb{Z}.$$

From linear independence, we have $x_1 = x_2$ and $y_1 = y_2$. Now given any positive integer n. Take a large enough circle centered at (α, β) so that its interior contains more than n lattice points. Since all the lattice points have different distances to (α, β), by shrinking the circle, we can remove lattice points one by one until we are left with n points.

We don't need such a strong independence assumption. Suppose $\alpha \notin \mathbb{Q}$ and $\beta \in \mathbb{Q}$. Then we have $x_1 = x_2$ and $(y_1 - y_2)(y_1 + y_2 - 2\beta) = 0$. So we just need $2\beta \notin \mathbb{Z}$.

9: (3 minutes)

Let $P(x)$ be a polynomial of degree m and let $Q(x)$ be a polynomial of degree n such that all the coefficients of P and Q are either 1 or 2022. If $P(x) \mid Q(x)$ as polynomials, then $m+1 \mid n+1$.

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T: Work in $R = \mathbb{Z}/2021\mathbb{Z}$. (If you prefer fields, work over $\mathbb{F}_{43}$ or $\mathbb{F}_{47}$.) Then $P(x) = (x^{m+1}-1)/(x-1)$ and $Q(x) = (x^{n+1}-1)/(x-1)$. Since $P(x) \mid Q(x)$ in $\mathbb{Z}[x]$, we have $x^{m+1}-1 \mid x^{n+1}-1$ in $R[x]$, which is only possible if $m+1 \mid n+1$.

Note we have

$$Q(x) = P(x)(1 + x^{m+1} + x^{2(m+1)} + \cdots + x^{n-m}) + 2021P(x)h(x)$$

for some polynomial $h(x)$. If $P(x) \neq 1 + x + \cdots + x^m$, does it follow that $h(x) = 0$?

10: (4 minutes)

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T: Given any 18-term arithmetic progression, the number of ways to color $\{1,2,\ldots,2022\}$ so that the given progression contains only one color is $2^{2022-17}$. The total number of 18-term arithmetic progressions is bounded by

$$\sum_{a=1}^{2022-18} \frac{2022-a}{17} = \frac{1}{17}\sum_{a=18}^{2021} a < \frac{2022\times 2021}{34} < \frac{2048\times 2048}{32} = 2^{17}.$$

Hence, the total number of ways to color $\{1,2,\ldots,2022\}$ so that some 18-term arithmetic progression contains only one color is less than the total number 2^{2022} of ways to color $\{1,2,\ldots,2022\}$. Therefore, there is way to color $\{1,2,\ldots,2022\}$ such that any 18-term arithmetic progressions contains both colors.

What is the biggest N such that the set $\{1,2,\ldots,2022\}$ can be colored with two colors such that any N-term arithmetic progressions contains both colors?

11: (3 minutes)

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F: Take $a_n = n!$. Then for any $k \leq n$, $k + n!$ is not a prime.

Can we make this true by limiting the growth rate of a_n? In other words, suppose A is a subset of $\mathbb{N}$. Suppose A has positive lower density:

$$\liminf_{n\to\infty} \frac{\#A \cap [1, n]}{n} > 0.$$

Does there exist k such that $k + A = \{k + a \colon a \in A\}$ contains infinitely many primes?

12: (4 minutes)

$$\int_0^{\pi} \ln\left(\frac{5}{4} - \cos x\right)\, dx > e^{-2022}.$$

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F: For any real number u, consider

$$I(u) = \int_0^\pi \ln(1 - 2u\cos x + u^2)\,dx.$$

Then the desired integral is $I(1/2)$. Since $\cos(\pi - x) = -\cos x$, we see that $I(u) = I(-u)$. Next note that

$$(1 - 2u\cos x + u^2)(1 + 2u\cos x + u^2) = 1 + 2u^2 + u^4 - 4u^2\cos^2 x = 1 + u^4 - 2u^2\cos 2x.$$

Hence

$$I(u) + I(-u) = \frac{1}{2}\int_0^{2\pi} \ln(1 - 2u^2\cos\theta + u^4)\,d\theta = \frac{1}{2}(I(u^2) + I(-u^2)) = I(u^2).$$

This implies that

$$I(u) = \frac{1}{2}I(u^2) = \frac{1}{4}I(u^4) = \cdots = \frac{1}{2^n}I(u^{2^n}).$$

Note if $0 \le u < 1$, then $\ln(1 - 2u\cos x + u^2) \le \ln((1+u)^2) < \ln 4$ and so $I(u) < \pi\ln 4$. The same is true for $I(u^{2^n})$ and so we have $I(u) = 0$ for $0 \le u < 1$.

$$I(u) = \int_0^\pi \ln(1 - 2u\cos x + u^2)\,dx, \qquad I(u) = I(-u) = \frac{1}{2}I(u^2).$$

From $I(u) = \frac{1}{2}I(u^2)$, we get also that $I(1) = 0$. For $u > 1$, we have

$$I(u) = I(1/u) + \int_0^\pi \ln(u^2)\,dx = 2\pi \ln u.$$

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T: Let p be the smallest prime divisor of $3^n - 2^n$. Then p is odd. Let $u = (p+1)/2$ so that $2u \equiv 1 \pmod{p}$. Hence $(3u)^n \equiv 1 \pmod{p}$. Let m denote the order of $3u$ in $\mathbb{F}_p^\times$. Then $m \mid \gcd(n, p-1)$. Since $3u \not\equiv 2u$, we have $m \neq 1$. Hence m has a prime divisor, which also divides n and is less than p.

This also proves the infinitude of primes.

14: (4 minutes)

$$\lim_{n\to\infty} \frac{n}{2^n} \sum_{k=1}^{n} \frac{2^k}{k} = 2.$$

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T: We note

$$\frac{n}{2^n}\sum_{k=1}^{n}\frac{2^k}{k} = \frac{n}{2^n}\sum_{k=0}^{n-1}\frac{2^{n-k}}{n-k} = \sum_{k=0}^{n-1}\frac{1}{2^k}\frac{n}{n-k} = \sum_{k=0}^{n-1}\frac{1}{2^k} + \sum_{k=0}^{n-1}\frac{1}{2^k}\frac{k}{n-k}.$$

The first sum tends to 2 as n goes to infinity. So it remains to prove the second sum goes to 0. We use the following bounds for $k/(n-k)$: for $k < \sqrt{n}$, $k/(n-k) < 2/\sqrt{n}$; for $\sqrt{n} \le k < n/2$, $k/(n-k) < 1$; and for $n/2 \le k < n$, $k/(n-k) < n$. Hence, we have

$$\sum_{k=0}^{n-1}\frac{1}{2^k}\frac{k}{n-k} \le \sum_{k=0}^{\lfloor\sqrt{n}\rfloor}\frac{1}{2^k}\frac{2}{\sqrt{n}} + \sum_{k=\lfloor\sqrt{n}\rfloor+1}^{\lfloor n/2\rfloor}\frac{1}{2^k} + \sum_{k=\lfloor n/2\rfloor+1}^{n}\frac{1}{2^k}n \ll \frac{1}{\sqrt{n}} + \frac{1}{2^{\sqrt{n}}} + \frac{n}{2^n} \to 0.$$

Since exponential functions grow much faster than polynomials, we can pretty much ignore the n and the k and just consider $\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=1}^{n}2^k$, which can be easily seen to be 2.

The same argument as above shows that if $P(x)$ is any polynomial, then

$$\lim_{n\to\infty} \frac{P(n)}{2^n} \sum_{k=1}^{n} \frac{2^k}{P(k)} = 2.$$

15: (4 minutes)

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T: We note that for a rectangle $D = [a, b] \times [c, d]$, up to a nonzero multiplicative constant, we have

$$\begin{aligned}
\iint_D \cos(2\pi x)\cos(2\pi y)\,dxdy &= (\sin(2\pi b) - \sin(2\pi a))(\sin(2\pi d) - \sin(2\pi c)),\\
\iint_D \cos(2\pi x)\sin(2\pi y)\,dxdy &= (\sin(2\pi b) - \sin(2\pi a))(\cos(2\pi d) - \cos(2\pi c)),\\
\iint_D \sin(2\pi x)\cos(2\pi y)\,dxdy &= (\cos(2\pi b) - \cos(2\pi a))(\sin(2\pi d) - \sin(2\pi c)),\\
\iint_D \sin(2\pi x)\sin(2\pi y)\,dxdy &= (\cos(2\pi b) - \cos(2\pi a))(\cos(2\pi d) - \cos(2\pi c)).
\end{aligned}$$

It is easy to see that all four integrals are 0 if and only if either $b - a \in \mathbb{Z}$ or $d - c \in \mathbb{Z}$. Since R can be tiled using rectangles with a side of integral length, the above four integrals over R is 0. Hence, R also has at least one side of integral length.

We are essentially trying to find a function $f(x)$, not necessarily continuous, such that $\int_a^b f(x)\,dx = 0$ if and only if $b - a \in \mathbb{Z}$. Such a function does not exist if it is real-valued. Indeed, take $F(x) = \int_0^x f(t)\,dt$. Then F is continuous and $F(0) = F(1) = 0$. Suppose F takes a maximum of $M > 0$ at $x = c$ on $[0, 1]$.

Then there exists $a \in (0, c)$ and $b \in (c, 1)$ such that $F(a) = F(b) = M/2$. Then $\int_a^b f(x)\,dx = 0$.

Such a function does exist over $\mathbb{C}$ by taking $f(x) = e^{2\pi i x}$, which is essentially what the solution uses.