Bernoulli Trial Problems for 2020

1: (2 minutes) If A is a proper subset of B and C is a proper subset of D, then $A \cup C$ is a proper subset of $B \cup D$.

F: Consider two disjoint sets A and B. Take $a \in A$ and $b \in B$. Then take $C = A \cup \{b\}$ and $D = B \cup \{a\}$.

2: (3 minutes) For all positive integers n, $\sqrt[n]{3} + \sqrt[n]{7} > \sqrt[n]{4} + \sqrt[n]{5}$.

T: The key observation here is $3 \times 7 = 21 > 20.25 = 4.5^2$. By the AM-GM inequality, $\sqrt[n]{3} + \sqrt[n]{7} > 2\sqrt[2n]{21} > 2\sqrt[n]{4.5}$. It then remains to check whether $\sqrt[n]{4.5} - \sqrt[n]{4} > \sqrt[n]{5} - \sqrt[n]{4.5}$. Consider the function $f(x) = \sqrt[n]{x+d} - \sqrt[n]{x}$ for some fixed $d > 0$. Then

$$f(x) = \frac{d}{(\sqrt[n]{x+d})^{n-1} + (\sqrt[n]{x+d})^{n-2}\sqrt[n]{x} + \cdots + (\sqrt[n]{x})^{n-1}}$$

is decreasing.

3: (3 minutes) There is an infinite set of positive integers such that no matter how we choose some elements of this set, their sum is not a perfect power.

T: Consider the set $A = \{2^n \cdot 3^{n+1} \colon n \geq 1\}$. Then the sum of any finite set of numbers from A is of the form $2^n \cdot 3^{n+1} \cdot y$ for some integer y congruent to 1 mod 6. Such a number cannot be a perfect power since the exponents of 2 and 3 are coprime.

4: (4 minutes) The Fourier transform of an integer $n \geq 9$ is defined to be the positive number b such that the base-b expansion of n has the most number of 4's among all expansions of n. If there is a tie among the bases with the most number of 4's, the average of these bases is the Fourier transform of n.

Among the integers from 10 to 99 inclusive, exactly four of them have Fourier transform 5.

T: Note first that any integer $n \geq 9$ equals 14 in base $n-4$. Secondly, to have Fourier transform 5, there must not be any ties. Hence we are left to consider integers from 10 to 99 whose base 5 representation contains 2 4's. These are $44_5 = 24$, $144_5 = 49$, $244_5 = 74$ and $344_5 = 99$. It remains to check they do not have 2 4's in other bases. Since they are all indivisible by 4, they are not of the form 44_b for any base b or 144_6 or 244_6. Finally $144_7 = 81$.

5: (3 minutes) There is a nine-digit perfect square, in which every digit except 0 appears and whose last digit is 5.

F: Suppose it exists. Then it is of the form $(10a+5)^2 = 100(a^2+a)+25$. Now the last digit of a^2+a is one of $0, 2, 6$. Only 6 is allowed and so our number is of the form $1000b + 625$ which is divisible by 125 and hence as a square, must be divisible by 625. So b is divisible by 5 but b cannot end in 0 or 5. Contradiction.

6: (3 minutes) There exists a set S of points in $\mathbb{R}^3$ such that for any plane P in $\mathbb{R}^3$, $P \cap S$ is finite and nonempty.

T: Consider the set $S = \{(t^5, t^3, t) \in \mathbb{R}^3 \colon t \in \mathbb{R}\}$. The intersection of S with any plane $Ax + By + Cz = D$ corresponds to the solutions to a polynomial of degree 5 or 3 or 1, which always has a real solution and at most 5 of them.

7: (3 minutes) The number 1280000401 is a prime.

F: It is pretty much impossible to show it is a prime given the time constraint, so it must be composite. Let $x = 20$. Then the given number is $x^7 + x^2 + 1$ which has the cube root of unity ζ_3 as a root since $\zeta_3^7 + \zeta_3^2 + 1 = \zeta_3 + \zeta_3^2 + 1 = 0$. Therefore, $x^7 + x^2 + 1$ is divisible by $x^2 + x + 1$. Hence the given number is divisible by 421. In fact, $x^7 + x^2 + 1 = (x^2 + x + 1)(x^5 - x^4 + x^2 - x + 1)$.

8: (3 minutes) There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x$ has 2019 solutions while $f(f(x)) = x$ has 9102 solutions.

F: Suppose such a function exist. Remove the 2019 solutions from $\mathbb{R}$ to get a set A. Restrict f to the set A. Then the equation $f(f(x)) = x$ has $9102 - 2019 = 7083$ solutions while the equation $f(x) = x$ has none. Hence the 7083 solutions to $f(f(x)) = x$ can be grouped in pairs (x_1, x_2) with the property that $f(x_1) = x_2$ and $f(x_2) = x_1$. This is impossible because 7083 is odd.

9: (3 minutes) It is possible to load a pair of dice (so they each have customized probabilities of rolling $1, 2, \ldots, 6$) so that the sum takes the values $2, 3, \ldots, 12$ equally likely.

F: Suppose it is possible. Let $p_1, \ldots, p_6$ and $q_1, \ldots, q_6$ denote the probabilities of rolling $1, \ldots, 6$ for the two dice. Then $p_1 q_1 = p_6 q_6 = 1/11$. Moreover,

$$\frac{1}{11} = p_1 q_6 + p_2 q_5 + \cdots + p_6 q_1 \geq p_1 q_6 + p_6 q_1 = p_1 q_1 (q_6/q_1) + p_6 q_6 (q_1/q_6) = \frac{1}{11}\left(\frac{q_6}{q_1} + \frac{q_1}{q_6}\right) \geq \frac{2}{11}.$$

Contradiction.

10: (4 minutes) If n is a positive integer, then $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor$.

T: First we check that $\sqrt{n} + \sqrt{n+1} \leq \sqrt{4n+2}$, which is equivalent to $2\sqrt{n(n+1)} \leq 2n+1$, which is equivalent to $4n^2 + 4n \leq 4n^2 + 4n + 1$, which is true. Since this is so tight, one can guess the statement is true. Suppose for a contradiction that there is an integer m such that $\sqrt{n} + \sqrt{n+1} < m \leq \sqrt{4n+2}$, which is equivalent to $2\sqrt{n(n+1)} < m^2 - 2n - 1 \leq 2n + 1$, which is equivalent to $4n^2 + 4n < (m^2 - 2n - 1)^2 \leq 4n^2 + 4n + 1$. This implies that $m^2 - 2n - 1 = 2n + 1$ and so $m^2 = 4n + 2$, which is congruent to 2 mod 4. Contradiction.

11: (4 minutes) There is a regular n-gon, where $n \neq 4$, all of those vertices are lattice points (i.e. the x- and y-coordinates of all the vertices are integers).

F: First we check that equilateral triangles do not exist. The area of an equilateral triangle is $\sqrt{3}\,a^2/4$ where a is the side length. The area of a potential equilateral triangle all of whose vertices are lattice points is then irrational, but the area of any triangle all of whose vertices are lattice points is rational. Next it is immediate that a hexagon cannot exist, for it contains an equilateral triangle. There is now a very cute infinite descent argument to show that no regular n-gon can exist for $n \neq 3, 4, 6$. Suppose such an n-gon exists with vertices $P_1, \ldots, P_n$. Then applying the vector P_2P_3 to P_1, the vector P_3P_4 to P_2, ..., the vector P_1P_2 to P_n, gives another regular n-gon with vertices that are lattice points, but is smaller!

12: (4 minutes) $\log_2 3 + \log_3 4 + \log_4 5 > 4$.

T: Set $a = \log_2 3$. Then $\log_3 4 = 2/a$. To compare $\log_4 5$ with a, we use the observation that $3 \cdot 2^{10} = 3072 < 3125 = 5^5$ and so $5 \log_2 5 > 10 + \log_2 3$. Hence we are left to consider the inequality

$$x + \frac{2}{x} + \frac{1}{2}\left(2 + \frac{x}{5}\right) > 4,$$

which is equivalent to $11x^2 - 30x + 20 > 0$. The roots of the quadratic equation are $(15 \pm \sqrt{5})/11$. The larger of the two roots is less than $(15 + 2.6)/11 = 1.6$. Finally we observe that $2^8 = 256 > 243 = 3^5$, which implies that $\log_2 3 > 8/5$.

In fact, $\log_2 3 + \log_3 4 + \log_4 5 = 4.007786...$

13: (4 minutes) A double-sequence of length n where n is a positive even number is a sequence in which $1, 2, 3 \ldots, n/2$ each appears twice, with the second occurrence of r being r positions after its first occurrence. For example, $4, 2, 3, 2, 4, 3, 1, 1$ is a double-sequence of length 8.

There is a double-sequence of length 2020.

F: Let $m = n/2$. For any $r = 1, \ldots, m$, let a_r denote the position of the first occurrence of r. Then the position of the second occurrence of r is $a_r + r$. The numbers $a_1, \ldots, a_m, a_1 + 1, \ldots, a_m + m$ then form a permutation of $1, \ldots, 2m$. Their sums must be equal. Let $A = a_1 + \cdots + a_m$. Then $2A + m(m+1)/2 = 2m(2m+1)/2$. So $A = (3m^2 + m)/4$. Only when $m \equiv 0, 1 \pmod 4$ is $3m^2 + m$ divisible by 4. In our case, $m = 1010$ is not 0 or 1 mod 4. Hence such a double-sequence does not exist.

14: (5 minutes) Suppose $f : [0, 1] \to \mathbb{R}$ is a function with continuous second derivative with $f(0) = f(1) = 0$ and that $f(x) > 0$ for all $x \in (0, 1)$. Then

$$\int_0^1 \left| \frac{f''(x)}{f(x)} \right| dx > 4.$$

T: Let $c \in (0, 1)$ so that $f(c) = M$ is the global maximum of $f(x)$ in $[0, 1]$. Then we have a first attempt:

$$\int_0^1 \left| \frac{f''(x)}{f(x)} \right| dx \geq \left| \int_0^1 \frac{f''(x)}{f(x)} \, dx \right| > \frac{1}{M} \left| \int_0^1 f''(x) \, dx \right| = \frac{1}{M} \big| f'(1) - f'(0) \big|.$$

However, we have no control over $f'(0)$ and $f'(1)$. The fix is now to take $a \in (0, c)$ such that $f'(a) = (f(c) - f(0))/(c - 0) = M/c$ and take $b \in (c, 1)$ such that $f'(b) = (f(c) - f(1))/(c - 1) = M/(c - 1)$.

Then we have

$$\begin{aligned}
\int_0^1 \left|\frac{f''(x)}{f(x)}\right| dx &\geq \int_a^b \left|\frac{f''(x)}{f(x)}\right| dx \geq \left|\int_a^b \frac{f''(x)}{f(x)}\, dx\right| \\
&> \frac{1}{M}\left|\int_a^b f''(x)\, dx\right| = \frac{1}{M}\left|f'(b) - f'(a)\right| \\
&= \left|\frac{1}{c-1} - \frac{1}{c}\right| \geq 4.
\end{aligned}$$

15: (5 minutes) Every closed convex region in the plane of area at least π contains two points whose distance from each other is at least 2.

T: Place the region on the plane so that it lies tangentially above the x-axis with the origin being the point of tangency. As a result, one may describe the region in polar coordinate as $\{(r,\theta)\colon 0 \leq r \leq f(\theta), 0 \leq \theta \leq \pi\}$ for some continuous function $f(x)$. Its area is then

$$\begin{aligned}
\int_0^\pi \int_0^{f(\theta)} r dr d\theta &= \frac{1}{2}\int_0^\pi f(\theta)^2 d\theta \\
&= \frac{1}{2}\left(\int_0^{\pi/2} f(\theta)^2 d\theta + \int_{\pi/2}^\pi f(\theta)^2 d\theta\right) \\
&= \frac{1}{2}\int_0^{\pi/2} \left(f(\theta)^2 + f(\theta+\pi/2)^2\right) d\theta.
\end{aligned}$$

Note that $f(\theta)^2 + f(\theta+\pi/2)^2$ is precisely the square of the distance from the point $(f(\theta),\theta)$ to the point $(f(\theta+\pi/2), \theta+\pi/2)$. Since the area of the region is at least π, some of these squares are at least 4.