Bernoulli Trials Problems for 2019

1: For every function $f : \mathbf{N} \to \mathbf{N}$ with $0 \le f(n) \le n$ for all $n \in \mathbf{N}$, the graph of f contains an infinite set of colinear points.

2: For $n = \prod_{i=1}^{l} {p_i}^{k_i}$ where $l \in \mathbf{Z}^+$, each $k_i \in \mathbf{Z}^+$ and p_i are distinct primes, let $f(n) = \sum_{i=1}^{l} k_i p_i$. Then $\sum_{n=2}^{\infty} \frac{1}{f(n)}$ converges.

3: For all integers $n \ge 3$, if $\varphi(n) = \varphi(n-1) + \varphi(n-2)$ then n is prime.

4: For every integer $n \ge 2$ there exists a nonzero $n \times n$ matrix A with entries in $\mathbf{Z}$ such that if we interchange any two rows in the matrix A then the resulting matrix B is skew-symmetric, that is $B^T = -B$.

5: There exists a sequence $\{a_n\}_{n\ge 1}$ where each $a_n \in \mathbf{R}^2$ with $a_n \to 0$ such that the open discs $D\big(a_n, \frac{1}{n}\big)$ are disjoint.

6: The closed unit square in $\mathbf{R}^2$ is equal to the union of a collection of disjoint sets each of which is homeomorphic to the open interval $(0,1)$.

7: There is a unique positive integer n such that there exists a connected planar graph G with n vertices each of which has degree 5.

8: For all $n, l \in \mathbf{Z}^+$, there exists a map $f : \mathbf{Z}_{n^l} \to \mathbf{Z}_n$ such that every sequence of length l in $\mathbf{Z}_n$ is of the form $f(k+1), f(k+2), \cdots, f(k+l)$ for some $k \in \mathbf{Z}_{n^l}$.

9: There exists an uncountable set S of subsets of $\mathbf{Z}$ with the property that for all $A, B \in S$ with $A \ne B$ the set $A \cap B$ is finite.

10: There exists a sequence of sets $A_1, A_2, A_3, \cdots$ where each A_n is an n-element set of positive real numbers with $\prod_{a \in A_n} a = 1$ such that $\lim_{n\to\infty} \left(\frac{1}{n} \sum_{a \in A_n} a \right) = 1$.

11: For every sequence $\{a_n\}_{n\ge 1}$ in $\mathbf{R}$, if $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = b \in \mathbf{R}$ and $\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{a_k}{k} = c \in \mathbf{R}$ then $b = c$.

12: $\displaystyle\int_0^\infty \ln^2\left(\frac{x}{x+3}\right) dx \ge 10.$

13: $1 + 6\cos\frac{2\pi}{7} \ge 2\sqrt{7}\,\cos\left(\frac{1}{3}\arctan 3\sqrt{3}\right).$

14: There exists a continuous function $f : [0,1] \to \mathbf{R}$ which crosses the x-axis at uncountably many points, where we say that f crosses the x-axis at a when $f(a) = 0$ and for all $\delta > 0$ there exist $x, y \in (a-\delta, a+\delta)$ with $f(x) < 0$ and $f(y) > 0$.

15: For $n \in \mathbf{Z}^+$ and $x \in \mathbf{R}$, define $f_n : \mathbf{R} \to [0,1)$ by $f_n(x) = nx - \lfloor nx \rfloor$. Then for some $a < b$, the sequence of functions $\{f_n : [a,b] \to \mathbf{R}\}$ has a convergent subsequence.