Solutions to the Bernoulli Trials Problems for 2018

1: There exists a positive integer k such that 2^k ends with the digits 2018 in its decimal representation.

Solution: This is FALSE. To have $2^k = 2018 \bmod 1000$ we would need $2^k = 2 \bmod 4$ but for $k \geq 2$ we have $2^k = 0 \bmod 4$.

2: There exists a positive integer n which is a multiple of 2018 such that the sum of the digits of n is equal to 2018.

Solution: This is TRUE. The sum of the digits of 2018 is 11 and the sum of the digits of 4036 is 13 and we have $11 \cdot 174 + 13 \cdot 8 = 2018$ so, for example, we can take n to be the number obtained by writing 174 copies of the digits 2018 followed by 8 copies of the digits 4036.

3: There exist infinitely many positive integers n such that $(2018n)!$ is a multiple of $n! + 1$.

Solution: This is FALSE. Recall that when $k_1, k_1, \cdots, k_\ell$ are non-negative integers with $k_1 + k_2 + \cdots + k_\ell = N$, the number $\frac{N!}{k_1! k_2! \cdots k_\ell!}$ is an integer and that for real numbers $a_1, a_2, \cdots, a_\ell$ we have

$$(a_1 + a_2 + \cdots a_\ell)^N = \sum_{k_1 + \cdots + k_\ell = N} \tfrac{N!}{k_1! k_2! \cdots k_\ell!} {a_1}^{k_1} \cdots {a_\ell}^{k_\ell}.$$

In particular, note that $(n!)^{2018} \big| (2018n)!$ and that we have

$$2018^{2018n} = (1 + 1 + \cdots + 1)^{2018n} = \sum_{k_1 + k_2 + \cdots + k_{2018} = 2018n} \tfrac{(2018n)!}{k_1! k_2! \cdots k_\ell!} > \tfrac{(2018n)!}{(n!)^{2018}}.$$

If $(n!+1) \big| (2018n)!$ then since $(n!)^{2018} \big| (2018n)!$ and $\gcd\big(n!+1, (n!)^{2018}\big) = 1$?? ???? $(n!+1)(n!)^{2018} \big| (2018n)!$ so that $(n!+1)(n!)^{2018} \leq (2018n)!$, and so

$$n! < n! + 1 \leq \frac{(2018n)!}{(n!)^{2018}} < 2018^{2018n} = \big(2018^{2018}\big)^n.$$

This can only be true for finitely many values of n since $\displaystyle\lim_{n\to\infty} \frac{(2018^{2018})^n}{n!} = 0$.

4: When $n = 2018$, there exists a permutation σ of the set $\{1, 2, \cdots, 3n-1, 3n\}$ with the property that $\sigma(3k) = \sigma(3k-1) + \sigma(3k-2)$ for all $k \in \{1, 2, \cdots, n\}$.

Solution: This is FALSE. If we had $\sigma(3k) = \sigma(3k-1) + \sigma(3k-2)$ for all $k \in \{1, 2, \cdots, n\}$ then we would have $\sigma(3k) + \sigma(3k-1) + \sigma(3k-2) = 2\sigma(3k)$ for all k and hence $\sigma(1) + \sigma(2) + \cdots + \sigma(3n)$ would be even. But when $n = 2018$, $\sigma(1) + \sigma(2) + \cdots + \sigma(3n) = 1 + 2 + \cdots + 3n = \frac{3n(3n+1)}{2} = \frac{3 \cdot 2018 \cdot (6054+1)}{2} = 3 \cdot 1009 \cdot 6055$, which is odd.

5: For all positive integers a and b with $\gcd(a, b) = 1$, there exist infinitely many positive integers k such that $a + kb$ is a Fibonacci number.

Solution: This is FALSE. Modulo 13, the first few Fibonacci numbers a_n are as follows:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_n	0	1	1	2	3	5	8	0	8	8	3	11	1	12	0	12

n	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
a_n	0	12	12	11	10	8	5	0	5	5	10	2	12	1	0	1

and then the pattern repeats every 28 terms with $a_{n+28} = a_n \bmod 13$. We see that for all n we have $a_n \neq 4$ and so there are no Fibonacci numbers of the form $a_n = 4 + 13k$.

6: For a polynomial of the form $f(x)=\sum_{k=0}^{20} c_k x^k$ with each $c_k\in\mathbf{Z}$ and $c_0=20$ and $c_{20}=3$, the largest possible number of distinct rational roots of $f(x)$ is equal to 6.

Solution: This is TRUE. Either all the rational roots of $f(x)$ are integers a with $a|20$, or at least one of the rational roots is of the form $x=\frac{a}{3}$ with $a|20$. When all the rational roots are integers, if the distinct rational roots are $a_1,a_2,\cdots,a_\ell$ then since $(x-a_1)(x-a_2)\cdots(x-a_\ell)\big|f(x)$ we see that $a_1a_2\cdots a_\ell\big|20$ so the largest possible number of distinct rational roots in this case is $\ell=5$ which occurs when the rational roots are $\{a_1,a_2,\cdots,a_5\}=\{1,-1,2,-2,\pm5\}$. When $f(x)$ has a rational root of the form $\frac{a_1}{3}$ where $a_1|20$, we have $f(x)=(3x-a_1)g(x)$ where $g(x)$ is monic with constant coefficient $-\frac{20}{a_1}$, so the other rational roots of $f(x)$ are all rational roots of $g(x)$ which must all be integers. In this case, if the rational roots of $f(x)$ are $\frac{a_1}{3},a_2,a_3,\cdots,a_\ell$ then since $(3x-a_1)(x-a_2)\cdots(x-a_\ell)\big|f(x)$ we see that $a_1a_2\cdots a_\ell\big|20$ so the largest possible number of distinct rational roots is $\ell=6$ which occurs when $\left\{\frac{a_1}{3},a_2,\cdots,a_6\right\}=\left\{\pm\frac{1}{3},1,-1,2,-2,\pm5\right\}$. One such polynomial is $f(x)=(3x-1)(x^2-1)(x^2-4)(x-5)(x^{14}+1)$.

7: There exists a bijective function from the Euclidean plane to the open unit disc which sends lines in the plane to chords in the disc.

Solution: This is FALSE. In the open disc, let $a=(0,0)$, $b=\left(\frac{1}{2},0\right)$, $c=\left(0,\frac{1}{2}\right)$, $d=\left(\frac{2}{3},\frac{1}{3}\right)$ and $e=\left(\frac{1}{3},\frac{2}{3}\right)$ and let A,B,C,D,E be the points in $\mathbf{R}^2$ which are mapped by $f(x)$ to the points a,b,c,d,e. The line AB is sent by $f(x)$ to the chord ab and the line AC maps to the chord ac. Note that C cannot lie on the line AB because c does not lie on the chord ab, and so the two lines AB and AC are not parallel and intersect at A. Since the line DE is sent by $f(x)$ to the chord de and the chord de does not intersect with either of the chords ab or ac in th open disc, it follows that the line DE cannot intersect with either of the lines AB or AC in $\mathbf{R}^2$. But this is not possible since the lines AB and AC are not parallel.

8: For every bounded function $f:\mathbf{R}\to\mathbf{R}$, if $f(x)+f\left(x+\frac{5}{6}\right)=f\left(x+\frac{1}{3}\right)+f\left(x+\frac{1}{2}\right)$ for all $x\in\mathbf{R}$ then f is periodic.

Solution: This is TRUE. Let $f(x)$ be bounded with $f(x)=f\left(x+\frac{1}{3}\right)+f\left(x+\frac{1}{2}\right)-f\left(x+\frac{5}{6}\right)$ for all x. Then for all x we have

$$\begin{aligned}
f(x)&=f\left(x+\tfrac{1}{3}\right)+f\left(x+\tfrac{1}{2}\right)-f\left(x+\tfrac{5}{6}\right)\\
&=\Big(f\left(x+\tfrac{2}{3}\right)+f\left(x+\tfrac{5}{6}\right)-f\left(x+\tfrac{7}{6}\right)\Big)+\Big(f\left(x+\tfrac{5}{6}\right)+f(x+1)-f\left(x+\tfrac{4}{3}\right)\Big)-f\left(x+\tfrac{5}{6}\right)\\
&=f\left(x+\tfrac{2}{3}\right)+f\left(x+\tfrac{5}{6}\right)-f\left(x+\tfrac{7}{6}\right)+f(x+1)-f\left(x+\tfrac{4}{3}\right)\\
&=\Big(f(x+1)+f\left(x+\tfrac{7}{6}\right)-f\left(x+\tfrac{3}{2}\right)\Big)+f\left(x+\tfrac{5}{6}\right)-f\left(x+\tfrac{7}{6}\right)+f(x+1)-f\left(x+\tfrac{4}{3}\right)\\
&=2f(x+1)-f\left(x+\tfrac{3}{2}\right)+f\left(x+\tfrac{5}{6}\right)-f\left(x+\tfrac{4}{3}\right)\\
&=2f(x+1)-f\left(x+\tfrac{3}{2}\right)+\Big(f\left(x+\tfrac{7}{6}\right)+f\left(x+\tfrac{4}{3}\right)-f\left(x+\tfrac{5}{3}\right)\Big)-f\left(x+\tfrac{4}{3}\right)\\
&=2f(x+1)-f\left(x+\tfrac{3}{2}\right)+f\left(x+\tfrac{7}{6}\right)-f\left(x+\tfrac{5}{3}\right)\\
&=2f(x+1)-f\left(x+\tfrac{3}{2}\right)+\Big(f\left(x+\tfrac{3}{2}\right)+f\left(x+\tfrac{5}{3}\right)-f(x+2)\Big)-f\left(x+\tfrac{5}{3}\right)\\
&=2f(x+1)-f(x+2)
\end{aligned}$$

and hence $f(x+2)-f(x+1)=f(x+1)-f(x)$. By induction, for all $x\in\mathbf{R}$ and all $k\in\mathbf{Z}^+$ we have $f(x+k)-f(x+k-1)=f(x+1)-f(x)$. Thus for $x\in\mathbf{R}$ and $n\in\mathbf{Z}^+$ we have

$$f(x+n)-f(x)=\sum_{k=1}^{n}\big(f(x+k)-f(x+k-1)\big)=n\big(f(x+1)-f(x)\big).$$

Since $f(x)$ is bounded, we must have $f(x+1)-f(x)=0$.

9: For all functions $u,v:\mathbf{R}\to\mathbf{R}$, if the function $f(x)=u\big(v(x)\big)$ is continuous then the function $u\big(-v(x)\big)$ is continuous.

Solution: This is FALSE. For example, if we define $u(x)$ by $u(x)=1$ for $x<0$ and $u(x)=0$ for $x\geq 0$ and if we define $v(x)$ by $v(x)=0$ for $x\leq 0$ and $v(x)=1$ for $x>0$ then we have $u\big(v(x)\big)=1$ for all x but $u\big(-v(x)\big)=0$ for $x\leq 0$ and $u\big(-v(x)\big)=1$ for $x>0$.

10: There exists a bounded $\mathcal{C}^\infty$ function $f : \mathbf{R} \to \mathbf{R}$ such that $\lim\limits_{n\to\infty} f^{(n)}(0) = \infty$.

Solution: This is TRUE. The function $g : (-1,1) \to \mathbf{R}$ given by $g(x) = \frac{1}{1-x}$ has derivatives $g^{(n)}(0) = n!$, so we can let $f(x) = g(x)h(x)$ for $|x| \le \frac{1}{2}$ and $f(x) = 0$ for $|x| \ge \frac{1}{2}$ where $h : \mathbf{R} \to [0,1]$ is a $\mathcal{C}^\infty$ bump function with $h(x) = 1$ for $|x| \le \frac{1}{4}$ and $h(x) = 0$ for $|x| \ge \frac{1}{2}$. Alternatively, we can let $f(x) = \sum\limits_{n=0}^{\infty} \frac{x^n}{\sqrt{n!}}$.

11: For every increasing function $f : (0,1) \to (0,1)$ with $f(x) > x$ for all $x \in (0,1)$, there exists a continuous function $g : (0,1) \to (0,1)$ which is not increasing and has the property that $g(x) < g\big(f(x)\big)$ for all $x \in (0,1)$.

Solution: This is TRUE. Let $f : (0,1) \to (0,1)$ be an increasing function with $f(x) > x$ for all $x \in (0,1)$. Let $a = f\big(\frac{1}{2}\big)$ and note that $a > \frac{1}{2}$. Define $g : (0,1) \to (0,1)$ by $g(x) = x$ for $x \in \big(0, \frac{1}{2}\big] \cup \big[a, 1\big)$ and $g(x) = h(x)$ for $x \in \big[\frac{1}{2}, a\big]$ where $h : \big[\frac{1}{2}, a\big] \to \big[\frac{1}{2}, a\big]$ is any continuous function which is not increasing such that $h\big(\frac{1}{2}\big) = \frac{1}{2}$ and $h(a) = a$ and $\frac{1}{2} < h(x) < a$ for all $x \in \big(\frac{1}{2}, a\big)$. If $0 < x < \frac{1}{2}$ and $f(x) \le \frac{1}{2}$ then we have $g\big(f(x)\big) = f(x) > x = g(x)$. If $0 < x < \frac{1}{2}$ and $f(x) > \frac{1}{2}$ then we have $f(x) \le f\big(\frac{1}{2}\big) = a$ so that $g\big(f(x)\big) = h\big(f(x)\big) \ge \frac{1}{2} > x = g(x)$. If $\frac{1}{2} \le x < a$ then we have $f(x) \ge f\big(\frac{1}{2}\big) = a$ so that $g\big(f(x)\big) = f(x) \ge a > h(x) = g(x)$. If $a \le x < 1$ then we have $f(x) > x \ge a$ so that $g\big(f(x)\big) = f(x) > x = g(x)$.

12: There exists a 4×4 real-valued matrix A such that $A^4 = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -2 \end{pmatrix}$.

Solution: This is FALSE. Suppose, for a contradiction, that A is such a matrix. Let $f_A(x)$ be the characteristic polynomial of A and let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the complex eigenvalues of A (that is the roots of $f_A(x)$, possibly with repetition). Then ${\lambda_1}^4$, ${\lambda_2}^4$, ${\lambda_3}^4$ and ${\lambda_4}^4$ are the reigenvalues of A^4 so, after possibly reordering the eigenvalues, we have ${\lambda_1}^4 = {\lambda_2}^4 = -1$ and ${\lambda_3}^4 = {\lambda_4}^4 = -2$. It follows that the eigenvalues cannot be real, and since the roots of $f_A(x)$ occur in conjugate pairs we must have $\lambda_2 = \overline{\lambda_1}$ and $\lambda_4 = \overline{\lambda_3}$. Thus the 4 complex eigenvalues of A are distinct, so the matrix A is diagonalizable over $\mathbf{C}$, so the matrix A^4 must also be diagonalizable over $\mathbf{C}$. But in fact A^4 is not diagonalizable over $\mathbf{C}$ since the eigenvalues -1 and -2 both have algebraic multiplicity 2, but the eigenspaces of -1 and of -2 are only 1-dimensional.

13: There exists a 2×2 integer-valued matrix A such that the entries of A^2 are prime numbers and the determinant of A is the square of a prime number.

Solution: This is TRUE. For example we can take $A = \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix}$ so that $A^2 = \begin{pmatrix} 2 & 11 \\ 11 & 101 \end{pmatrix}$.

14: There exists a decreasing sequence of positive real numbers $\{a_n\}$ such that $\sum\limits_{n=1}^{\infty} a_n$ diverges and $\sum\limits_{n=1}^{\infty} n!\,a_{n!}$ converges.

Solution: This is TRUE. Let $a_n = \frac{1}{n \cdot \log n \cdot \log(\log n)}$ for $n \ge 3$ and choose $a_1 > a_2 > a_3$ so that the sequence $\{a_n\}$ is positive and decreasing. Note that the series $\sum\limits_{n=1}^{\infty} a_n$ diverges by the integral test. For all $0 \le k < n$ we have $(k+1)(n-k) = (k+1)n - (k+1)k \le (k+1)n - nk = n$, and so

$$(n!)^2 = (1 \cdot n)(2 \cdot (n-1))(3 \cdot (n-2)) \cdots (n \cdot 1) \ge n \cdot n \cdot n \cdots n = n^n.$$

It follows that $2\log(n!) = \log((n!)^2) \ge \log(n^n) = n \log n$ and hence $\log(n!) \ge \frac{n \log n}{2}$. When $n \ge 8 > e^2$ we also have $\frac{n \ln n}{2} > n$ so that $\log(n!) > n$, and so

$$n!\,a_{n!} = \frac{1}{\log(n!) \cdot \log(\log(n!))} > \frac{1}{\frac{n \log n}{2} \cdot \log n} = \frac{2}{n(\log n)^2}.$$

The series $\sum\limits_{n=8}^{\infty} \frac{2}{n(\log n)^2}$ converges by the integral test, so the series $\sum\limits_{n=1}^{\infty} n!\,a_{n!}$ converges by comparison.

15: There exists a sequence of complex numbers $\{a_n\}$ with the property that for all positive integers p, the sum $\sum\limits_{n=1}^{\infty} a_n{}^p$ converges if and only if p is a prime number.

Solution: This is TRUE. Let P be the set of prime numbers. Let n be a positive integer. Choose complex numbers $w_1, w_2, \cdots, w_n$ such that for all $1 \leq p \leq n$ we have

$$\sum_{k=1}^{n} w_k{}^p = \begin{cases} 0 \text{ if } p \in P, \\ 1 \text{ if } p \notin P \end{cases}$$

(the polynomial with roots $w_1, w_2, \cdots, w_n$ has coefficients which are symmetric polynomials in $w_1, \cdots, w_n$ and can hence be expressed in terms of the sums $\sum\limits_{k=1}^{n} w_k{}^p$). Choose $m = m_n \geq 1$ larger than the maximum value of $\sum\limits_{k=1}^{r} \left|w_k{}^p\right|$ over all choices of $1 \leq r \leq n$ and $p \in P$ with $1 \leq p \leq n$. Let $\ell = \ell_n = n(nm)^n$ and let $(a_{n,1}, a_{n,2}, \cdots, a_{n,\ell})$ be the sequence obtained by repeating the sequence $\left(\frac{w_1}{nm}, \frac{w_2}{nm}, \cdots, \frac{w_n}{nm}\right)$ a total of $(nm)^n$ times. For all $1 \leq p \leq n$ we have

$$\sum_{k=1}^{\ell} a_{n,k}{}^p = (nm)^n \sum_{k=1}^{\ell} \left(\tfrac{w_k}{nm}\right)^p = (nm)^{n-p} \sum_{k=1}^{n} w_k{}^p = \begin{cases} \quad 0 & \text{if } p \in P\,, \\ (nm)^{n-p} & \text{if } p \notin P\,, \end{cases}$$

and when $p \in P$ and $1 \leq s \leq \ell$, if we write $s = qn + r$ with $0 \leq r < n$ we have

$$\left|\sum_{k=1}^{s} a_{n,k}{}^p\right| = \left|\sum_{k=1}^{r} \left(\tfrac{w_k}{nm}\right)^p\right| \leq \tfrac{m}{(nm)^p} \leq \tfrac{1}{n}.$$

Let $(a_1, a_2, a_3, \cdots)$ be the sequence $(a_{1,1}, a_{1,2}, \cdots, a_{1,\ell_1}.a_{2,1}, \cdots, a_{2,\ell_2}, a_{3,1}, \cdots, a_{3,\ell_3}, \cdots)$. Then for all positive integers p, if $p \in P$ then $\sum\limits_{k=1}^{\infty} a_k{}^p$ converges and if $p \notin P$ then $\sum\limits_{k=1}^{\infty} a_k{}^p$ diverges.