Solutions to the Bernoulli Trials Problems for 2017

1: For all integers $n \geq 2$, we have $n = 3 \bmod 6$ if and only if $n^2 + 2^n$ is prime.

Solution: This is FALSE. When $n = 27$ we have $n^2 + 2^n = 27^2 + 2^{27} = 9^3 + 512^3 = (9+512)(9^2 - 9\cdot 512 + 512^2)$ which is not prime.

2: For all real numbers x and y with $2 \leq x \leq y$, we have $y^{x+1} \leq x\, y^y$.

Solution: This is TRUE. When $x = y$ we have $y^{x+1} = x^{x+1} = x\, x^x = x\, y^y$. Suppose that $2 \leq x < y$. Note that $y^{x+1} \leq x\, y^y \iff (x+1)\ln y \leq \ln x + y \ln y \iff \ln y - \ln x \leq (y-x)\ln y \iff \frac{\ln y - \ln x}{y-x} \leq \ln y$. By the Mean Value Theorem we choose $c \in (x,y)$ so that $\frac{\ln y - \ln x}{y-x} = \frac{1}{c}$, and then $\frac{\ln y - \ln x}{y-x} = \frac{1}{c} < \frac{1}{x} \leq \frac{1}{2} < \ln 2 < \ln y$, as required.

3: For every sequence $\{a_n\}_{n\geq 1}$ of real numbers, if $\sum\limits_{n=1}^{\infty} a_n$ converges then $\sum\limits_{n=1}^{\infty} {a_n}^3$ converges.

Solution: This is FALSE. For example, consider the sequence $\frac{2}{\sqrt[3]{1}}, \frac{-1}{\sqrt[3]{1}}, \frac{-1}{\sqrt[3]{1}}, \frac{2}{\sqrt[3]{2}}, \frac{-1}{\sqrt[3]{2}}, \frac{-1}{\sqrt[3]{2}}, \frac{2}{\sqrt[3]{3}}, \frac{-1}{\sqrt[3]{3}}, \frac{-1}{\sqrt{3}}, \cdots$

4: There exist 6 open discs in $\mathbf{R}^2$ such that each disc contains the point $(0,0)$ and no disc contains another disc's centre

Solution: This is FALSE. Suppose, for a contradiction, that we have 6 such discs. Since the intersection is nonempty and open, we can choose a point p in the intersection which is not equal to any of the 6 centres. Consider the 6 line segments from this point to each of the centres. The angle between some adjacent 2 segments is less than or equal to $60°$. But then the centre which is at the end of the shorter of these two segments lies inside the disc whose centre lies at the end of the longer of the two segments.

5: There exist 99 lines in $\mathbf{R}^2$ such that for all $k, l \in \{1, 2, \cdots, 100\}$, one of the lines passes through the interior of the square with vertices at (k,l), $(k-1,l)$, $(k-1,l-1)$ and $(k,l-1)$.

Solution: This is TRUE. We can use the 98 lines of slope $-\frac{1}{2}$ with equations $2x + 4y = 3 + 6k$ for $1 \leq k \leq 98$ together with the line of slope $\frac{n-1}{n-2}$ with equation $2(n-1)x + 2(n-2)y = n$.

6: For every positive integer n there exist matrices $A, B \in M_n(\mathbf{R})$ such that

$$\big\{X \in M_n(\mathbf{R})\big|AX = XA \text{ and } BX = XB\big\} = \big\{cI\big|c \in \mathbf{R}\big\}.$$

Solution: This is TRUE. When $n = 1$ we can take $A = B = 0$ and when $n \geq 2$ we can take A to be the matrix with entries $A_{k,l} = 1$ when $l = k+1$ and $A_{k,l} = 0$ when $l \neq k+1$, and we can take the matrix B to be the matrix $B = A^T$. Verify that for $X \in M_n(\mathbf{R})$ we have $AX = XA$ when X is upper triangular with equal entries along each diagonal (that is X is of the form $X = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{n-1}A^{n-1}$ for some $c_i \in \mathbf{R}$) and similarly $XB = BX$ when X is lower diagonal with equal entries along each diagonal.

7: When we rearrange the alternating harmonic series so that each positive term is followed by 4 negative terms, as shown below, the resulting series converges and its sum is zero.

$$\tfrac{1}{1} - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{3} - \tfrac{1}{10} - \tfrac{1}{12} - \tfrac{1}{14} - \tfrac{1}{16} + \tfrac{1}{5} - \tfrac{1}{18} - \cdots = 0.$$

Solution: This is TRUE. Let S_n denote the n^{th} partial sum of the given series. Since the terms in the series tend to zero, it suffices to show that the partial sums S_{5n} tend to 0. Recall that $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$. Also note that $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{4n}$ is equal to the Riemann sum for the function $f(x) = \frac{1}{x}$ on the interval $[1,4]$ using $3n$ equal-sized subintervals. Thus we have

$$\begin{aligned} S_{5n} &= \left(\tfrac{1}{1} + \tfrac{1}{3} + \tfrac{1}{5} + \cdots + \tfrac{1}{2n-1}\right) - \left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \cdots + \tfrac{1}{8n}\right) \\ &= \left(\tfrac{1}{1} - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots + \tfrac{1}{2n-1} - \tfrac{1}{2n}\right) - \left(\tfrac{1}{2n+2} + \tfrac{1}{2n+4} + \tfrac{1}{2n+6} + \cdots + \tfrac{1}{8n}\right) \\ &= \left(\tfrac{1}{1} - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots + \tfrac{1}{2n-1} - \tfrac{1}{2n}\right) - \tfrac{1}{2}\left(\tfrac{1}{n+1} + \tfrac{1}{n+2} + \tfrac{1}{n+3} + \cdots + \tfrac{1}{4n}\right) \end{aligned}$$

so that $\lim\limits_{n\to\infty} S_{5n} = \ln 2 - \frac{1}{2}\displaystyle\int_1^4 \frac{1}{x}\, dx = \ln 2 - \frac{1}{2}\ln 4 = 0.$

8: There exists a bijection $f:\mathbf{Z}^+\to\mathbf{Z}^+$ such that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+f(n)}$ converges.

Solution: This is TRUE. It is quite challenging to find such a bijection. One such bijection is as follows.

	$0!^2$	$1!^2$		$2!^2$						$3!^2$						$4!^2$			$5!^2$
k	1	2	3	4	5	6	$\cdots$	34	35	36	37	$\cdots$	546	547	$\cdots$	576	577	$\cdots$	$5!^2$
$f(k)$	4	36	35	1	576	575	$\cdots$	547	3	2	$5!^2$	$\cdots$	c	34	$\cdots$	5	$6!^2$		37

where $c=5!^2-509$. To be precise, for $n!^2<x\le(n+1)!^2$, we define $f(x)$ by

$$\begin{aligned} f\big(n!^2+k\big)&=(n+2)!^2-(k-1)\ ,\ \text{for } 1\le k\le a_n\\ f\big((n+1)!^2-k\big)&=(n-1)!^2+(k+1)\ ,\ \text{for } 0\le k< b_n\end{aligned}$$

where $a_0=1$, and $b_0=0$, and for $n\ge1$, $b_n=a_{n-1}$ and $a_n+b_n=(n+1)!^2-n!^2$. Under this bijection we have

$$\begin{aligned}\sum_{x=n!^2+1}^{(n+1)!^2}\frac{1}{x+f(x)}&=\sum_{k=1}^{a_n}\frac{1}{n!^2+k+f\big(n!^2+k\big)}+\sum_{k=0}^{b_n-1}\frac{1}{(n+1)!^2-k+f\big((n+1)!^2-k\big)}\\ &=\sum_{k=1}^{a_n}\frac{1}{n!^2+(n+2)!^2+1}+\sum_{k=0}^{b_n-1}\frac{1}{(n+1)!^2+(n-1)!^2+1}\\ &=\frac{a_n}{n!^2+(n+2)!^2+1}+\frac{b_n}{(n+1)!^2+(n-1)!^2+1}\\ &<\frac{(n+1)!^2-n!^2}{n!^2+(n+2)!^2+1}+\frac{n!^2-(n-1)!^2}{(n+1)!^2+(n-1)!^2+1}\\ &<\frac{(n+1)!^2}{(n+2)!^2}+\frac{n!^2}{(n+1)!^2}=\frac{1}{(n+2)^2}+\frac{1}{(n+1)^2}\,.\end{aligned}$$

Since $\sum\frac{1}{(n+2)^2}$ and $\sum\frac{1}{(n+1)^2}$ converge, it follows that $\sum\frac{1}{n+f(n)}$ converges.

9: There exists a bijection $f:\mathbf{Z}^+\to\mathbf{Z}^+$ such that for every $n\in\mathbf{Z}^+$ we have $n\,\Big|\,\sum\limits_{k=1}^{n}f(k)$.

Solution: This is TRUE. We construct such a sequence recursively as follows. Let $a_2=1$ and $a_1=3$. Having chosen $a_1,a_2,\cdots,a_{2n}$, choose a_{2n+2} to be the smallest positive integer with $a_{2n+2}\notin\{a_1,a_2,\cdots,a_{2n}\}$ then choose a_{2n+1} to be equal to the smallest positive integer x with $x\notin\{a_1,a_2,\cdots,a_{2n}\}\cup\{a_{2n+2}\}$ which satisfies $x=-(a_1+a_2+\cdots+a_{2n}) \mod 2n+1$ and $x\equiv-(a_1+a_2+\cdots+a_{2n})-a_{2n+2} \mod 2n+2$ (note that the integer x exists by the Chinese Remainder Theorem).

10: There exist disjoint nonempty homeomorphic sets A and B with $A\cup B=[0,1]$.

Solution: This is TRUE. The interval $[0,1]$ is homeomorphic to the extended real line $\mathbf{R}\cup\{\pm\infty\}$ which can be partitioned into the disjoint nonempty homeomorphic sets $C=\Big(\bigcup_{k\in\mathbf{Z}}[2k,2k+1)\Big)\cup\{\infty\}$ and $D=\Big(\bigcup_{k\in\mathbf{Z}}[-2k-1,-2k)\Big)\cup\{-\infty\}$. We can obtain a homeomorphism from C to D by sending ∞ to $-\infty$ and, for each $k\in\mathbf{Z}$, sending the interval $[2k,2k+1)$ homeomorphically to the interval $[-2k-1,-2k)$.

11: Define $F:\mathbf{Z}[x]\to[0,1]$ as follows. Given a polynomial $f\in\mathbf{Z}[x]$, let $F(f)$ be the real number x with decimal representation $x=0.a_1a_2a_3\cdots$ where $a_k\in\{0,1,2,\cdots,9\}$ with $a_k=f(k)\bmod 10$. Then the range of F contains less than 12,536 elements.

Solution: This is TRUE. Note that for all k we have $f(k+2)=f(k)\bmod 2$, $f(k+5)=f(k)\bmod 5$ and $f(k+10)\equiv f(k)\bmod 10$. Since $f(k+10)=f(k)\bmod 10$, the sequence $\{a_k\}$ is periodic with period 10. Since $f(k+2)=f(k)\bmod 2$, the terms a_0, a_2, a_4, a_6 and a_8 all have the same parity and the terms a_1, a_3, a_5, a_7 and a_9 all have the same parity, so there are 4 ways to choose the parities. Having chosen the parities, there are 5 choices for each of the terms $a_0,a_1,\cdots,a_4$ and then the chosen parities together with the condition that $f(k+5)\equiv f(k)\bmod 5$ uniquely determine the terms $a_5,a_6,\cdots,a_9$. Thus the total number of choices for the terms $a_0,a_1,\cdots,a_9$ is at most $4\cdot5^5=12{,}500$.

12: There exists a 20-element set $S \subseteq \mathbf{Z}_{210}$ such that $S+S=\mathbf{Z}_{210}$.

Solution: This is FALSE. Let $S=\{a_1,a_2,\cdots,a_{20}\}$. In order to have $S+S=\mathbf{Z}_{210}$, each of the 210 one- or two-element sets $\{j,k\}$ with $1\le j,k\le 20$ must yield a different sum $a_j+a_k\in\mathbf{Z}_{210}$. There are $20\cdot 19=380$ ordered pairs (j,k) with $1\le j,k\le 20$ and $j\ne k$. Choose two distinct such ordered pairs (j,k) and (l,m) such that $a_k-a_j=a_m-a_l\in\mathbf{Z}_{210}$. Then we have $a_j+a_m=a_k+a_l\in\mathbf{Z}_{210}$. Since $j\ne k$ and $l\ne m$ and $(j,k)\ne(l,m)$ it follows that $\{j,m\}\ne\{k,l\}$ and so $S+S\ne\mathbf{Z}_{210}$.

13: Five regular tetrahedra with unit side length can be arranged in space so that they all share a common edge and are otherwise disjoint.

Solution: This is TRUE. Say the vertices of the tetrahedron are at A, B, C and D (draw a picture). The angle θ between the faces ABC and BCD is equal to the angle between the altitude L of the face ABC from the vertex A to the midpoint E of BC and the altitude M of the face BCD from D to E. The altitude H of the tetrahedron from the vertex D meets the base ABC at its centroid G which lies $\frac{2}{3}$ of the way along the base altitude L from A to E. The triangle DGE is a right angle triangle with angle θ at vertex E and we see that $\cos\theta=\frac{1}{3}$. On the other hand, recall (or show) that $\cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4}$. Thus we have $\theta<\frac{2\pi}{5}\iff\frac{1}{3}>\frac{\sqrt{5}-1}{4}\iff 4>3\sqrt{5}-3\iff 7>3\sqrt{5}\iff 49>45$, which is true, so indeed $\theta<\frac{2\pi}{5}$.

14: Nine points can be arranged on the unit sphere so that each of the 9 points has exactly 4 equidistant nearest neighbours.

Solution: This is TRUE. Place 3 of the points around the circle $z=0$ at positions $\big(1,0,0\big)$ and $\big(-\frac{1}{2},\pm\frac{\sqrt{3}}{2},0\big)$, then choose c with $0<c<1$, set $r=\sqrt{1-c^2}$, and place 3 points along each of the two circles at $z=\pm c$ at positions $\big(-r,0,\pm c\big)$ and $\big(\frac{1}{2}r,\pm\frac{\sqrt{3}}{2}r,c\big)$. It is intuitive that it is possible to choose c so that each of the 9 points has 4 nearest neighbours. A calculation (which we omit) shows that we need to choose $c=\frac{\sqrt{5}}{3}$, and so the 9 points lie at $\big(1,0,0\big)$, $\big(-\frac{1}{2},\pm\frac{\sqrt{3}}{2},0\big)$, $\big(-\frac{2}{3},0,\pm\frac{\sqrt{5}}{3}\big)$ and $\big(\frac{1}{3},\pm\frac{\sqrt{3}}{3},\pm\frac{\sqrt{5}}{3}\big)$. It is easy to check that the Euclidean distance between nearest neighbours is then equal to $\frac{2}{\sqrt{3}}$.

15: There exist infinitely many primes p with the property that for all $a,b\in\mathbf{Z}^+$ with $a<b$ and $\gcd(a,b)=1$ such that a, b and p are distinct modulo p, the exponent of p in the prime factorization of $b^{p-1}-a^{p-1}$ is odd.

Solution: This is FALSE. For any odd prime p, if we let $a=p^2-2$ and $b=p^2+2$ then we have $a<b$, and $\gcd(a,b)=1$, and a, b and c are distinct modulo p, and since $2^{p-1}\equiv 1 \bmod p$ it is easy to check that the exponent of p in the prime factorization of $b^{p-1}-a^{p-1}$ is equal to 2.