Solutions to the Bernoulli Trials Problems for 2016

1: There are exactly 10! seconds in 6 weeks.

Solution: This is TRUE. The number of seconds in 6 weeks is $6\cdot 7\cdot 24\cdot 60\cdot 60=6\cdot 7\cdot 8\cdot 3\cdot 10\cdot 3\cdot 2\cdot 5\cdot 4\cdot 3=10!$.

2: The product of any three consecutive integers, the middle of which is a perfect cube, is a multiple of 504.

Solution: This is TRUE. Let the middle of the three integers be x^3 so their product is $(x^3-1)(x^3)(x^3+1)=x^3(x^6-1)$. By Fermat's Little Theorem, when $x\neq 0 \mod 7$ we have $x^6=1 \mod 7$, and so for all x we have $x^3(x^6-1)=0 \mod 7$. By the Euler-Fermat Theorem, when $x\neq 0 \mod 9$ we have $x^6=1 \mod 9$ and so for all x we have $x^3(x^6-1)=0 \mod 9$. When x is even we have $x^3=0 \mod 8$ and when x is odd we have $x^2=1 \mod 8$ hence $x^6=1 \mod 8$, and so for all x we have $x^3(x^6-1)=0 \mod 8$. Since $x^3(x^6-1)=0$ modulo 7, 8 and 9, we have $x^3(x^6)=0 \mod 504$ by the Chinese Remainder Theorem.

3: The number 6 is the only squarefree perfect number.

Solution: This is TRUE. Let n be squarefree and perfect. Say $n=p_1p_2\cdots p_l$ where the p_i are distinct primes. Note that $l>1$ since no prime is perfect, and we have If n was odd then since $\sigma(n)=\prod_{i=1}^l(p_i+1)$ with $l>1$ we would have $4\big|\sigma(n)$ hence $4|2n$ so that $2|n$. Thus n is even. Since n is an even perfect number, we have $n=2^{p-1}M_p$ for some Mersenne prime $M_p=2^p-1$. Since $n=2^{p-1}M_p$ is squarefree, we must have $p=2$ and so $n=2\cdot M_2=2\cdot 3=6$.

4: The only positive integer solution to the equation $x^2+7=y^3$ is $(x,y)=(1,2)$.

Solution: This is FALSE. For example $(x,y)=(181,32)$ is a solution.

5: For all nonzero rational numbers a and b, if $c=\frac{ab}{a+b}$ then $\sqrt{a^2+b^2+c^2}$ is rational.

Solution: This is TRUE. Indeed $(a+b-c)^2=\big((a+b)-\frac{ab}{a+b}\big)^2=(a+b)^2-2ab+c^2=a^2+b^2+c^2$.

6: $\tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7}\leq 20$.

Solution: This is FALSE. In fact the sum is equal to 21. Recall that

$$\begin{aligned}\cos 7\theta+i\sin 7\theta&=(\cos\theta+i\sin\theta)^7\\&=\Big(\tbinom{7}{0}\cos^7\theta-\tbinom{7}{2}\cos^5\theta\sin^2\theta+\cdots\Big)+i\Big(\tbinom{7}{1}\cos^6\theta\sin\theta-\tbinom{7}{3}\cos^4\theta\sin^3\theta+\cdots\Big)\end{aligned}$$

so we have

$$\begin{aligned}\sin\theta\sin 7\theta&=\tbinom{7}{1}\cos^6\theta\sin^2\theta-\tbinom{7}{3}\cos^4\theta\sin^4\theta+\tbinom{7}{5}\cos^2\theta\sin^6\theta-\tbinom{7}{7}\sin^8\theta\\&=\cos^8\theta\Big(\tbinom{7}{1}\tan^2\theta-\tbinom{7}{3}\tan^4\theta+\tbinom{7}{5}\tan^6\theta-\tbinom{7}{7}\tan^8\theta\Big)\\&=\cos^8\theta\,\big(7-35\tan^2\theta+21\tan^4\theta-\tan^6\theta\big).\end{aligned}$$

When $k\in\{1,2,3\}$ and $\theta=\frac{k\pi}{7}$, we have $\sin 7\theta=0$ and $\cos\theta\neq 0$, so the above formula shows that $\tan^2\theta$ is a root of he polynomial $f(x)=x^3-21x^2+35x-7$. Thus the three numbers $\alpha=\tan^2\frac{\pi}{7}$, $\beta=\tan^2\frac{2\pi}{7}$ and $\gamma=\tan^2\frac{3\pi}{7}$ are the three distinct roots of $f(x)$ and so we have $\alpha+\beta+\gamma=21$ (and also $\alpha\beta\gamma=7$).

We remark that, more generally, a similar argument shows that

$$\sum_{i=1}^n\tan^2\tfrac{k\pi}{2n+1}=\tbinom{2n+1}{2}=2n^2+n\quad\text{and}\quad\prod_{k=1}^n\tan^2\tfrac{k\pi}{2n+1}=2n+1.$$

7: $\sqrt{e}<\pi^2/6$.

Solution: This is FALSE. Indeed

$$\pi^2<(3.142)^2=9.872164<9.88\ \text{ and}$$
$$6\,e^{1/2}>6\big(1+\tfrac12+\tfrac{1}{2!}\left(\tfrac12\right)^2+\tfrac{1}{3!}\left(\tfrac12\right)^3+\tfrac{1}{4!}\left(\tfrac12\right)^4\big)=6+3+\tfrac34+\tfrac18+\tfrac{1}{64}=9.890625>9.89.$$

8: For every positive integer n and every matrix $A \in M_n(\mathbf{C})$ which is not a constant multiple of the identity matrix, the vector space $U = \big\{X \in M_n(\mathbf{C}) \big| AX = XA\big\}$ is spanned by the set $\{I, A, A^2, A^3, \cdots\}$.

Solution: This is FALSE. For example, if A is the 3×3 matrix with a 1 in the top left corner and all other entries equal to 0, then U is the space of block-diagonal matrices with a 1×1 block in the upper left and a 2×2 block in the lower right, so $\dim(U) = 5$, but $\text{Span}\{I, A, A^2, \cdots\} = \text{Span}\{I, A\}$ which is 2-dimensional.

9: Two players, A and B, take turns, beginning with A, filling in the entries of a 25×25 matrix with real numbers. Player A wins if the final matrix is not invertible and player B wins if it is invertible. In this game, player B has a winning strategy.

Solution: This is FALSE. It is player A that has a winning strategy as follows. Player A begins by placing any entry in column 13, the centre column (leaving an even number of entries in that column). Each time B places an entry in the centre column, A does the same. Each time player B places an entry x in column $k \neq 13$, A places the entry x in column $26 - k$. The final matrix is symmetric about the centre column.

10: There exists a bijective map $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{n^2}$ converges.

Solution: This is FALSE. Let $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ be bijective. Given $m \in \mathbf{Z}^+$, we have

$$\sum_{n=m+1}^{3m} f(n) \geq 1 + 2 + \cdots 2m = m(2m+1) > 2m^2$$

and so

$$\sum_{n=m+1}^{3m} \frac{f(n)}{n^2} \geq \sum_{n=m+1}^{3m} \frac{f(n)}{(3m)^2} = \frac{1}{9m^2} \sum_{n=m+1}^{3m} f(n) > \tfrac{2m^2}{9m^2} = \tfrac{2}{9}\,.$$

11: There exists a bijective map $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\, f(n)}$ diverges.

Solution: This is FALSE. Note that if $j < k$ but $f(j) > f(k)$ then we have

$$\left(\tfrac{1}{j\,f(k)} + \tfrac{1}{k\,f(j)}\right) - \left(\tfrac{1}{j\,f(j)} + \tfrac{1}{k\,f(k)}\right) = \left(\tfrac{1}{j} - \tfrac{1}{k}\right)\left(\tfrac{1}{f(k)} - \tfrac{1}{f(j)}\right) > 0$$

and hence

$$\tfrac{1}{1\cdot f(1)} + \tfrac{1}{2\cdot f(2)} + \cdots + \tfrac{1}{j\,f(j)} + \cdots \tfrac{1}{k\,f(k)} + \cdots \tfrac{1}{n\,f(n)} < \tfrac{1}{1\cdot f(1)} + \tfrac{1}{2\cdot f(2)} + \cdots + \tfrac{1}{k\,f(k)} + \cdots \tfrac{1}{j\,f(j)} + \cdots \tfrac{1}{n\,f(n)}\,.$$

It follows that, for $l \in \mathbf{Z}^+$, if $\big\{f(1), f(2), \cdots, f(l)\big\} = \big\{a_1, a_2, \cdots, a_l\big\}$ with $a_1 < a_2 < \cdots < a_l$ then

$$S_l = \sum_{n=1}^{l} \tfrac{1}{n\,f(n)} = \tfrac{1}{1\cdot f(1)} + \tfrac{1}{2\cdot f(2)} + \cdots + \tfrac{1}{l\cdot f(l)} \leq \tfrac{1}{1\cdot a_1} + \tfrac{1}{2\cdot a_2} + \cdots + \tfrac{1}{l\cdot a_l} \leq \tfrac{1}{1\cdot 1} + \tfrac{1}{2\cdot 2} + \cdots + \tfrac{1}{l\cdot l} = \sum_{n=1}^{l} \tfrac{1}{n^2}\,.$$

Since the sum $\sum \frac{1}{n^2}$ converges, the sequence S_l is bounded above by $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, and so it converges.

12: $\displaystyle\int_0^{\pi/2} \tan x \big|\ln(\sin x)\big|\, dx > \tfrac{\pi}{8}$.

Solution: This is TRUE. Note first that, for any positive integer m, we have

$$\int_0^1 t^m \ln t\; dt = \Big[\tfrac{1}{m+1}\, t^{m+1} \ln t\Big]_0^1 - \int_0^1 \tfrac{1}{m+1}\, t^m\, dt = 0 - \Big[\tfrac{1}{(m+1)^2}\, t^{m+1}\Big]_0^1 = -\tfrac{1}{(m+1)^2}.$$

Let $u = \sin x$ so that $du = \cos x\, dx$. Then using the fact that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ we have

$$\begin{aligned}\int_0^{\pi/2} \tan x \big|\ln(\sin x)\big|\, dx &= -\int_{x=0}^{\pi/2} \tan x \ln(\sin x)\, dx = -\int_{u=0}^{1} \frac{u \ln u}{1-u^2}\, du = -\int_0^1 \sum_{k=0}^{\infty} u^{2k+1} \ln u\; du \\ &= \sum_{k=0}^{\infty} -\int_0^1 u^{2k+1} \ln u\; du = \sum_{k=0}^{\infty} \tfrac{1}{(2k+2)^2} = \tfrac{1}{4} \sum_{n=1}^{\infty} \tfrac{1}{n^2} = \tfrac{\pi^2}{24} > \tfrac{3\pi}{24} = \tfrac{\pi}{8}.\end{aligned}$$

13: The function $f:\mathbf{R}\to\mathbf{R}$ given by $f(0)=0$ and $f(x)=\sin\frac{1}{x}$ for $x\neq 0$ has an antiderivative.

Solution: This is TRUE. Indeed if we let $F(0)=0$ and $F(x)=x^2\cos\frac{1}{x}-\displaystyle\int_0^x 2t\cos\frac{1}{t}\,dt$ for $x\neq 0$ then it is fairly easy to check that $F'(x)=f(x)$ for all x.

14: Let $f_1(x)=x$ and $f_{n+1}(x)=x^{f_n(x)}$ for $n\geq 1$. Then the function $g(x)=\lim\limits_{n\to\infty}\frac{1}{f(x)}$ is continuous for $x>1$.

Solution: This is FALSE. We claim that $g(x)$ is discontinuous at $e^{1/e}$. First note that for $x>1$ the sequence $1/f_n(x)$ is positive and decreasing with n so $g(x)$ is well-defined. We claim that $g(e^{1/e})\geq 1/e$. Note that $f_1(e^{1/e})=e^{1/e}\leq e$. Let $n\in\mathbf{Z}^+$ and suppose, inductively, that $f_n(e^{1/e})\leq e$. Then $f_{n+1}(e^{1/e})=(e^{1/e})^{f_n(e^{1/e})}\leq(e^{1/e})^e=e$. It follows, by induction, that $f_n(e^{1/e})\leq e$ for all $n\geq 1$, and hence $g(e^{1/e})\geq 1/e$.

To prove that $g(x)$ is discontinuous at $e^{1/e}$, we shall show that $g(x)=0$ for all $x>e^{1/e}$. Let $x>e^{1/e}$. Note that since the sequence $\{f_n(x)\}$ is increasing, either it converges to some finite number a, in which case $g(x)=\frac{1}{a}>0$, or it diverges to infinity, in which case $g(x)=0$. Suppose, for a contradiction, that the sequence $\{f_n(x)\}$ converges to a finite number, say $\lim\limits_{n\to\infty}f_n(x)=a$. Note that since the sequence increases with $f_1(x)=x>1$ we must have $a>1$. We have

$$a=\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}f_{n+1}(x)=\lim_{n\to\infty}x^{f_n(x)}=x^{\lim_{n\to\infty}f_n(x)}=x^a.$$

hence $x=a^{1/a}$. For the function $y(t)=t^{1/t}$ we have $\frac{dy}{dt}/y=(1-\ln t)/t^2$, so that $y(t)$ attains its maximum when $t=e$. It follows that $x=a^{1/a}\leq e^{1/e}$, which gives the desired contradiction.

15: In the symmetric group S_5, the identity element is equal to the composite of the 10 distinct transpositions, listed in some order.

Solution: This is TRUE. For example, $e=(45)(34)(35)(23)(14)(24)(13)(25)(12)(15)$.

16: There exists a function $f:\mathbf{R}\to\mathbf{R}$ such that f is differentiable in a dense set $A\subseteq\mathbf{R}$ and f is discontinuous in a dense set $B\subseteq\mathbf{R}$.

Solution: This is TRUE. For example, we can define $f(x)=0$ whenever x is *not* of the form $x=\frac{k}{2^n}$ where $n\in\mathbf{Z}^+$ and k is an odd integer, and define $f\left(\frac{k}{2^n}\right)=\frac{1}{4^n}$ where $n\in\mathbf{Z}^+$ and k is an odd integer. Then f is discontinuous at every point of the form $\frac{k}{2^n}$ with $n\in\mathbf{Z}^+$ and k odd, and f is differentiable with derivative zero at every point of the form $\frac{l}{3^m}$ with $m\in\mathbf{Z}^+$ and $\gcd(l,3)=1$ (and also at some other points) because to have $\frac{k}{2^n}\to\frac{l}{3^m}$ we must have $n\to\infty$ since $\left|\frac{k}{2^n}-\frac{l}{3^m}\right|\geq\frac{1}{2^n3^m}$ and because

$$\left|\frac{f\left(\frac{k}{2^n}\right)-f\left(\frac{l}{3^m}\right)}{\frac{k}{2^n}-\frac{l}{3^m}}\right|=\frac{\frac{1}{4^n}}{\left|\frac{k}{2^n}-\frac{l}{3^m}\right|}\leq\frac{\frac{1}{4^n}}{\frac{1}{2^n3^m}}=\frac{3^m}{2^n}\to 0\text{ as }n\to\infty.$$

17: The set of rational numbers is equal to the disjoint union of countably many sets, each of which is dense in the set of real numbers.

Solution: This is TRUE. Here is one example. Let $b_1<b_2<b_3<\cdots$ be the sequence of positive integers which are *not* perfect powers (so for example, $b_1=2$, $b_2=3$ and $b_3=5$). Let $S_1=\left\{\frac{a}{2^k}\middle|k\in\mathbf{Z}^+,a\in\mathbf{Z}\right\}$ and for $n\geq 2$ let $S_n=\left\{\frac{a}{{b_n}^k}\middle|k\in\mathbf{Z}^+,a\in\mathbf{Z},\gcd(a,b_n)=1\right\}$. Notice that every rational number lies in one of the sets S_n (with $\mathbf{Z}$ being contained in S_1) and that the sets S_n are disjoint. Also, each set S_n is dense in $\mathbf{R}$ because every closed interval of length $\frac{1}{{b_n}^k}$ contains a number in S_n of the form $\frac{cb_n+1}{{b_n}^{k+1}}$.

18: Ann and Bob each flip a coin 10 times. The probability that Ann and Bob flip the same number of heads as each other is greater than $\frac{1}{6}$.

Solution: This is TRUE. The probabilty that Ann and Bob flip the same number of heads is

$$P = \sum_{k=0}^{10} \left(\frac{\binom{10}{k}}{2^{10}} \right)^2 = \frac{1}{2^{20}} \sum_{k=0}^{10} \binom{10}{k}^2 = \frac{1}{2^{20}} \binom{20}{10} = \frac{1}{2^{20}} \frac{20 \cdot 19 \cdot 18 \cdots 11}{10 \cdot 9 \cdot 8 \cdots 1} = \frac{19 \cdot 17 \cdot 13 \cdot 11}{2^{18}}$$

and so we have

$$\begin{aligned} P > \tfrac{1}{6} &\iff 19 \cdot 17 \cdot 13 \cdot 11 \cdot 3 > 2^{17} \iff 19 \cdot 13 \cdot (16+1) \cdot (32+1) > 2^{17} \\ &\iff 247(16 \cdot 32 + 49) > 2^{17} \iff 247 \cdot 512 + 247 \cdot 49 > 256 \cdot 512 \\ &\iff 247 \cdot 49 > 9 \cdot 512 \end{aligned}$$

and this is clearly true since $247 \cdot 49 > 200 \cdot 40 = 8000$ while $9 \cdot 512 < 10 \cdot 600 = 6000$.