Bernoulli Trials Problems for 2016

1: There are exactly 10! seconds in 6 weeks.

2: The product of any three consecutive integers, the middle of which is a perfect cube, is a multiple of 504.

3: The number 6 is the only squarefree perfect number.

4: The only positive integer solution to the equation $x^2 + 7 = y^3$ is $(x, y) = (1, 2)$.

5: For all nonzero rational numbers a and b, if $c = \frac{ab}{a+b}$ then $\sqrt{a^2 + b^2 + c^2}$ is rational.

6: $\tan^2 \frac{\pi}{7} + \tan^2 \frac{2\pi}{7} + \tan^2 \frac{3\pi}{7} \leq 20$.

7: $\sqrt{e} < \pi^2/6$.

8: For every positive integer n and every matrix $A \in M_n(\mathbf{C})$ which is not a constant multiple of the identity matrix, the vector space $U = \left\{X \in M_n(\mathbf{C}) \middle| AX = XA\right\}$ is spanned by the set $\{I, A, A^2, A^3, \cdots\}$.

9: Two players, A and B, take turns, beginning with A, filling in the entries of a 25×25 matrix with real numbers. Player A wins if the final matrix is not invertible and player B wins if it is invertible. In this game, player B has a winning strategy.

10: There exists a bijective map $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{n^2}$ converges.

11: There exists a bijective map $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\, f(n)}$ diverges.

12: $\displaystyle\int_0^{\pi/2} \tan x \big| \ln(\sin x) \big| \, dx > \frac{\pi}{8}$.

13: The function $f : \mathbf{R} \to \mathbf{R}$ given by $f(0) = 0$ and $f(x) = \sin \frac{1}{x}$ for $x \neq 0$ has an antiderivative.

14: Let $f_1(x) = x$ and $f_{n+1}(x) = x^{f_n(x)}$ for $n \geq 1$. Then the function $g(x) = \lim\limits_{n \to \infty} \frac{1}{f(x)}$ is continuous for $x > 1$.

15: In the symmetric group S_5, the identity element is equal to the composite of the 10 distinct transpositions, listed in some order.

16: There exists a function $f : \mathbf{R} \to \mathbf{R}$ such that f is differentiable in a dense set $A \subseteq \mathbf{R}$ and f is discontinuous in a dense set $B \subseteq \mathbf{R}$.

17: The set of rational numbers is equal to the disjoint union of countably many sets, each of which is dense in the set of real numbers.

18: Ann and Bob each flip a coin 10 times. The probability that Ann and Bob flip the same number of heads as each other is greater than $\frac{1}{6}$.