Solutions to the Bernoulli Trials Problems for 2013

1: $1+1+1-1+1+1-1+1-1-1+1+1+1-1+1+1-1+1-1-1+1-1-1+1-1-1-1+1+1-1-1+1-1-1+1+1+1+1-1+1-2+1-1+1-1-1=0\,.$

Solution: This is TRUE. If you got this one wrong, then you should review your notes from Grade 1.

2: There exist $a,b\in\mathbf{Q}$ such hat $a^b\in\mathbf{Q}$ but $b^a\notin\mathbf{Q}$.

Solution: This is TRUE. For example we can take $a=\frac{1}{2}$ and $b=2$.

3: It is possible to partition $\mathbf{Q}^+$ into two non-empty disjoint sets which are each closed under addition.

Solution: This is FALSE. Suppose $\mathbf{Q}^+=A\cup B$ where A and B are non-empty and closed under addition. Say $\frac{a}{b}\in A$ and $\frac{c}{d}\in B$ where $a,b,c,d\in\mathbf{Z}^+$. Then $ac=bc\cdot\frac{a}{b}\in A$ and $ac=ad\cdot\frac{c}{d}\in B$ and so $A\cap B\neq\emptyset$.

4: For all positive integers a, b and n, if $a|n$ and $b|n$ and $ab<n$ then $\gcd\left(\frac{n}{a},\frac{n}{b}\right)>1$.

Solution: This is TRUE. Suppose that $a|n$, $b|n$, and $\gcd\left(\frac{n}{a},\frac{n}{b}\right)=1$. Choose integers x and y so that $\frac{n}{a}\,x+\frac{n}{b}\,y=1$. Multiply by ab to get $n(x+y)=ab$. Since $a,b,n>0$ this implies $x+y>0$, so $x+y\geq 1$, and so $ab=n(x+y)\geq n$.

5: For all continuous functions $f:\mathbf{R}\to\mathbf{R}$, if for every $0\neq c\in\mathbf{R}$ the graph of $y=cf(x)$ is congruent to the graph of $y=f(x)$, then $f(x)=ax+b$ for some $a,b\in\mathbf{R}$. (Two sets are *congruent* when they are related by an isometry, that is a composite of translations, rotations and reflections).

Solution: This is FALSE. For example the function $f(x)=e^x$ has this property.

6: For $n\in\mathbf{Z}^+$, let a_n be the number of congruence classes of triangles with integer sides and perimeter n. Then for every odd integer $n\in\mathbf{Z}^+$ we have $a_n=a_{n+3}$.

Solution: This is TRUE. Note that $a_n=|A_n|$ where A_n is the set of ordered triples (a,b,c) with $a\leq b\leq c$, $a+b>c$ and $a+b+c=n$. For all $n\in\mathbf{Z}^+$, the map $\phi:A_n\to A_{n+3}$ given by $\phi(a,b,c)=(a+1,b+1,c+1)$ is clearly injective. Note that if $(1,b,c)\in A_m$ then since $b\leq c$ and $1+b>c$ we have $b=c$, and so $m=1+b+c=1+2b$, which is odd. Thus when n is odd, so that $n+3$ is even, there are no ordered triples of the form $(1,b,c)$ in A_{n+3}, and so the map ϕ is bijective with inverse given by $\phi^{-1}(a,b,c)=(a-1,b-1,c-1)$.

7: There exists a continuous map $f:[0,1]\to[0,\pi]$ such that f restricts to a bijection $f:\mathbf{Q}\cap[0,1)\to\mathbf{Q}\cap[0,\pi)$.

Solution: This is TRUE. Let $a_n=1-\frac{1}{2^n}$ so that we have $0=a_0<a_1<a_2<\cdots$ with $\lim\limits_{a_n}=1$, and choose a sequence of rational numbers $\{b_n\}$ with $0=b_0<b_1<\cdots$ such that $\lim\limits_{n\to\infty}b_n=\pi$. Then define $f:[0,1]\to[0,\pi]$ by

$$f(x)=b_n+\frac{b_n-b_{n-1}}{a_n-a_{n-1}}(x-a_n)\text{ for }a_{n-1}\leq x\leq a_n\,.$$

8: There exists a twice-differentiable function $f:\mathbf{R}\to\mathbf{R}$ with $f''(0)\neq 0$ with the property that $f'(x)=f(x+1)-f(x)$ for all $x\in\mathbf{R}$.

Solution: This is TRUE. We look for a solution $y=f(x)$ to the differential equation $f'(x)=f(x+1)-f(x)$ of the form $f(x)=e^{rx}$. When $f(x)=e^{rx}$ we have $f'(x)=re^{rx}$ and $f(x+1)-f(x)=e^{rx+r}-e^{rx}=e^{rx}(e^r+1)$, so in order for $f(x)=e^{rx}$ to be a solution, we need $r=e^r-1$. There is clearly no real solution, so we consider $r=s+it$ with $s,t\in\mathbf{R}$. Then $f(x)=e^{rx}$ is a solution when $r=e^r-1$, that is when $s+it=e^se^{it}-1$, or equivalently when $s=e^s\cos t-1$ (1) and $t=e^s\sin t$ (2). From equation (2) we have $e^s=\frac{t}{\sin t}$ so $s=\ln(t)-\ln(\sin t)$. We put this into equation (1) to get $\ln t-\ln(\sin t)=\frac{t}{\sin t}\cos t-1$, or equivalently $g(t)=0$ where $g(t)=\big(\ln t-\ln(\sin t)+1\big)\sin t-t\cos t$. Since $\lim\limits_{t\to 2\pi^+}g(t)=2\pi$ and $\lim\limits_{t\to 3\pi^-}=-3\pi$, it follows from the Intermediate Value Theorem that we can choose $t\in(2\pi,3\pi)$ so that $g(t)=0$. We then choose $s=\ln(t)-\ln(\sin t)$ so that equations (1) and (2) hold. Then $f(x)=e^rx=e^{(s+it)x}$ is a complex solution to the differential equation. The real and imaginary parts of f, that is the functions $u(x)=e^{sx}\cos(tx)$ and $v(x)=e^{sx}\sin(tx)$, are both real solution to the differential equation.

9: For all $n \in \mathbf{Z}^+$ and for all $n \times n$ matrices A and B, we have $e^{A+B} = e^A e^B$. (For an $n \times n$ matrix X, $e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots$).

Solution: This is FALSE. For example, let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Note that $A = A^2 = A^3 = \cdots$ and $0 = B^2 = B^3 = \cdots$ and so we have

$$e^A = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}\ ,\ e^B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } e^A e^B = \begin{pmatrix} e & e \\ 0 & 1 \end{pmatrix}.$$

On the other hand, let $C = A + B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Note that for $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ we have $Cu = u$ and $Cv = 0$. For $P = (u, v)$, we have $CP = (u, 0) = PA$ so $C = PAP^{-1}$. Thus

$$e^{A+B} = e^C = e^{PAP^{-1}} = Pe^A P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix}.$$

10: There exists a 3×3 matrix A over $\mathbf{Z}_2$ with $A \neq I$ and $A^7 = I$.

Solution: This is TRUE. For such a matrix A, the minimal polynomial of A is of degree at most 3 and divides the polynomial $x^7 - 1 = (x-1)(x^6 + x^5 + \cdots + 1) = (x+1)(x^3 + x + 1)(x^3 + x^2 + 1)$. We can take A to be the companion matrix of the polynomial $x^3 + x + 1$, that is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

11: The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1+1/\sqrt{\ln n}}}$ converges.

Solution: This is TRUE. Indeed we have $n^{1+1/\sqrt{\ln n}} = n \cdot n^{1/\sqrt{\ln n}} = n \cdot e^{\ln n/\sqrt{\ln n}} = n \cdot e^{\sqrt{\ln n}}$, and so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1+1/\sqrt{\ln n}}} = \sum_{n=2}^{\infty} \frac{1}{n \cdot e^{\sqrt{\ln n}}}$, which converges by the Integral Test since, letting $u = \sqrt{\ln x}$ so that $u^2 = \ln x$ and $2u\,du = \frac{1}{x}\,dx$, we have

$$\int_1^\infty \frac{dx}{x\,e^{\sqrt{\ln x}}} = \int_{u=0}^\infty 2u\,e^{-u}\,du = \Big[-2(u+1)e^{-u}\Big]_0^\infty = 2\,.$$

12: The series $\displaystyle\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{k^2 + l^2}$ converges.

Solution: This is FALSE. We have

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{k^2 + l^2} \geq \int_{x=1}^\infty \int_{y=1}^\infty \frac{1}{x^2 + y^2}\,dy\,dx = \int_{x=0}^\infty \left[\tfrac{1}{x}\ \tan^{-1}\tfrac{y}{x}\right]_1^\infty dx = \int_{x=1}^\infty \tfrac{1}{x}\left(\tfrac{\pi}{2} - \tan^{-1}\tfrac{1}{x}\right) dx = \infty$$

because $\displaystyle\int_1^\infty \tfrac{1}{x} \cdot \tfrac{\pi}{2}\,dx = \infty$ while, since $\tan^{-1} u \leq u$ for all $u > 0$, we have $\displaystyle\int_1^\infty \tfrac{1}{x} \cdot \tan^{-1}\tfrac{1}{x}\,dx \leq \int_1^\infty \tfrac{1}{x^2}\,dx = 1.$

13: The sequence $\dfrac{1}{(\ln n)^2}\displaystyle\sum_{k=1}^{n}\left(\sqrt[k]{k}-1\right)$ converges as $n\to\infty$.

Solution: This is TRUE, and the limit is equal to $\frac{1}{2}$. We can see this somewhat informally as follows. We have $\sqrt[k]{k}-1=k^{1/k}-1=e^{\ln k/k}-1=\frac{\ln k}{k}+\frac{1}{2!}\left(\frac{\ln k}{k}\right)^2+\frac{1}{3!}\left(\frac{\ln k}{k}\right)^3+\cdots\sim\frac{\ln k}{k}$ and so

$$\sum_{k=1}^{n}\left(\sqrt[k]{k}-1\right)\sim\sum_{k=1}^{n}\frac{\ln k}{k}\sim\int_1^n\frac{\ln x}{x}\,dx=\tfrac{1}{2}(\ln n)^2\,.$$

We can make this rigorous as follows. For $u=\frac{\ln x}{x}$ with $x\geq 1$ we have $u'=\frac{1-\ln x}{x^2}$, so u attains its maximum at $x=e$ and we have $u\leq\frac{1}{e}<\ln 2$ for all $x\geq 1$. By Taylor's Theorem, since $0\leq u\leq\ln 2$ we have $0\leq\frac{e^0}{2}\,u^2\leq e^u-(1+u)\leq\frac{e^{\ln 2}}{2}\,u^2=u^2$ and so $u\leq e^u-1\leq u+u^2$. Thus we have

$$\frac{\ln x}{x}\leq e^{\ln x/x}-1\leq\frac{\ln x}{x}+\left(\frac{\ln x}{x}\right)^2$$

$$\sum_{k=1}^{n}\frac{\ln k}{k}\leq\sum_{k=1}^{n}\left(\sqrt[k]{k}-1\right)\leq\sum_{k=1}^{n}\left(\frac{\ln k}{k}+\frac{(\ln k)^2}{k^2}\right)$$

$$a+\int_1^n\frac{\ln x}{x}\,dx\leq\sum_{k=1}^{n}\left(\sqrt[k]{k}-1\right)\leq b+\int_1^n\frac{\ln x}{x}-\frac{(\ln x)^2}{x^2}\,dx$$

$$a+\left[\tfrac{1}{2}(\ln x)^2\right]_1^n\leq\sum_{k=1}^{n}\left(\sqrt[k]{k}-1\right)\leq b+\left[\tfrac{1}{2}(\ln x)^2-\tfrac{(\ln x)^2}{x}-\tfrac{2\ln x}{x}-\tfrac{2}{x}\right]_1^n$$

$$a+\tfrac{1}{2}(\ln n)^2\leq\sum_{k=1}^{n}\left(\sqrt[k]{k}-1\right)\leq b+\tfrac{1}{2}(\ln n)^2-\tfrac{(\ln n)^2}{n}-\tfrac{2\ln n}{n}-\tfrac{2}{n}+2\,.$$

Divide all three terms by $(\ln n)^2$ then use the Squeeze Theorem to see that $\displaystyle\lim_{n\to\infty}\frac{1}{(\ln n)^2}\sum_{k=1}^{n}\left(\sqrt[k]{k}-1\right)=\tfrac{1}{2}$.

14: The series $\displaystyle\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^{kn}$ converges.

Solution: This is TRUE. We claim that $\left(\frac{k}{n}\right)^k\leq\frac{1}{2}$ for all such k,n. Let $f(x)=\left(\frac{x}{n}\right)^x=e^{x\ln(x/n)}$ for $1\leq x\leq n-1$. Then $f'(x)=e^{x\ln(x/n)}\left(\ln\frac{x}{n}+1\right)$ so we have $f'(x)<0$ for $x<\frac{n}{e}$ and $f'(x)>0$ for $x<\frac{n}{e}$. Thus f attains its minimum at $x=\frac{n}{e}$ and it attains its maximum at one of the two endpoints. We have $f(1)=\frac{1}{n}$ and $f(n-1)=\left(\frac{n-1}{n}\right)^{n-1}$. Since $n\geq 2$ we have $f(1)=\frac{1}{n}\leq\frac{1}{2}$. To prove our claim, we shall show that $\left(\frac{n-1}{n}\right)^{n-1}$ decreases from $\frac{1}{2}$ for $n\geq 2$ (in fact it decreases towards $\frac{1}{e}$ as $n\to\infty$). Let

$$g(x)=\left(\tfrac{x-1}{x}\right)^{x-1}=\left(1-\tfrac{1}{x}\right)^{x-1}=e^{(x-1)\ln\left(1-\frac{1}{x}\right)}$$

for $x>1$. Then we have

$$g'(x)=e^{(x-1)\ln\left(1-\frac{1}{x}\right)}\left(\ln\left(1-\tfrac{1}{x}\right)+(x-1)\cdot\frac{\frac{1}{x^2}}{1-\frac{1}{x}}\right)=\left(\tfrac{x-1}{x}\right)^{x-1}\left(\ln\left(1-\tfrac{1}{x}\right)+\tfrac{1}{x}\right)<0$$

for all $x>1$ because $\ln(1-u)+u<0$ for all $u<1$. Thus $\left(\frac{n-1}{n}\right)^{n-1}$ decreases from $\frac{1}{2}$ for $n\geq 2$, and hence we have proven our claim that $\left(\frac{k}{n}\right)^k\leq\frac{1}{2}$ for all $n\geq 2$ and $1\leq k\leq n-1$. It follows that

$$\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{kn}=\sum_{k=1}^{n}\left(\left(\tfrac{k}{n}\right)^k\right)^n\leq\sum_{k=1}^{n}\left(\tfrac{1}{2}\right)^n=\frac{n}{2^n}\,.$$

We know that $\sum\frac{n}{2^n}$ converges, and so the given sum converges by comparison.

15: Euclidean space can be partitioned into a disjoint union of pairwise skew lines.

Solution: This is TRUE. For each $r > 0$ and $\theta \in [0, 2\pi)$, let $L_{r,\theta}$ be the line given by

$$(x, y, z) = \big(r\cos\theta, r\sin\theta, 0\big) + t\big(-r\sin\theta, r\cos\theta, 1\big)\,.$$

The lines $L_{r,\theta}$ together with the y-axis form such a partition. Indeed, for fixed $r > 0$, when the line $L_{r,0}$ is revolved about the z-axis, it sweeps out the hyperboloid H_r given by $x^2 + y^2 - r^2z^2 = r^2$, so the hyperboloid H_r is partitioned by the lines $L_{r,\theta}$, and Euclidean space is partitioned by these hyperboloids H_r together with the z-axis.