Solutions to the Bernoulli Trials Problems for 2012

1: There exists a positive integer n such that $n^3 + (n+1)^3 = (n+2)^3$.

Solution: This is FALSE. Replace n by $x-1$ and let $f(x) = (x-1)^3 + x^3 - (x+1)^3 = x^3 - 6x^2 - 2$. The only possible rational roots of f are $x = \pm 1, \pm 2$ but $f(-2) = -34$, $f(-1) = -9$, $f(1) = -7$ and $f(2) = -16$ so f has no rational roots.

2: There exists a positive integer n such that neither n nor n^2 uses the digit 1 in its base 3 representation.

Solution: This is FALSE. Indeed if n does not use the digit 1 in its base 3 representation, then its base 3 representation ends with $200\cdots 0$ (say with $0 \leq k$ zeros) and so the base 3 representation of n^2 ends with $100\cdots 0$ (with $2k+1$ zeros).

3: For every positive integer n, n is prime if and only if there exist unique positive integers a and b such that $\frac{1}{n} = \frac{1}{a} - \frac{1}{b}$.

Solution: This is TRUE. For any positive integer n, we have $\frac{1}{n} = \frac{1}{n-1} - \frac{1}{n(n-1)}$ so we can take $a = n-1$ and $b = n(n-1)$. When $n = kl$ with $k, l > 1$, we also have $\frac{1}{n} = \frac{1}{k(l-1)} - \frac{1}{k\,l(l-1)}$, so in this case a and b are not uniquely determined. Suppose n is prime. To get $\frac{1}{n} = \frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$ we need $n(b-a) = ab$. Since n is prime, we have $n\big|a$ or $n\big|b$. If we had $n\big|a$ with say $a = kn$ then we would have $n(b-a) = ab = knb$ so that $b - a = nb$, but this is not possible since $b - a < b$ while $nb \geq b$. Thus $n \not\big| a$ and hence $n\big|b$, say $b = ln$. We have $n(b-a) = ab = aln \Longrightarrow b - a = al \Longrightarrow b = (l+1)a \Longrightarrow ln = (l+1)a$. Since n is prime and $n \not\big| a$ we have $n\big|(l+1)$, and since $\gcd(l, l+1) = 1$ we have $l\big|a$. Say $(l+1) = sn$ and $a = tl$. Then $ln = (l+1)a = stln$ so $s = t = 1$ and we have $l + 1 = n$ and $a = l = n - 1$. Thus in the case that n is prime, the values of a and hence b are uniquely determined.

4: $\sqrt{1 + \sqrt{7 + \sqrt{1 + \sqrt{7 + \cdots}}}}$ is rational.

Solution: This is TRUE. Let $l = \sqrt{1 + \sqrt{7 + \sqrt{1 + \sqrt{7 + \cdots}}}}$, assuming that this expression is well defined. Then we have $l^2 = 1 + \sqrt{7 + \sqrt{1 + \sqrt{7 + \cdots}}}$, hence $l^4 - 2l^2 + 1 = 7 + l$, that is $l^4 - 2l^2 - l - 6$. Thus l is a root of $f(x) = x^4 - 2x^2 - x - 6$. The only possible rational roots of f are $x = \pm 1, \pm 2, \pm 3, \pm 6$. We try some of these and find that $f(2) = 0$, then long division gives $f(x) = (x-2)g(x)$ where $g(x) = (x^3 + 2x^2 + 2x + 3)$. We have $g'(x) = 3x^2 + 4x + 2$, which has negative discriminant, so $g'(x) > 0$ for all x. Thus g is increasing with $g(0) = 3$, so g has 1 real root which is negative. Since 2 is the only positive real root of f, it follows that $l = 2$ (assuming that l is well-defined).

We will not bother to verify that l is well-defined. We remark that one way to define the number l rigorously is to define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_{n+1} = \sqrt{1 + \sqrt{7 + a_n}}$ for $n \geq 1$, then verify that $\{a_n\}$ converges and let $l = \lim\limits_{n \to \infty} a_n$.

5: $\sin(20^\circ)\sin(40^\circ)\sin(60^\circ)\sin(80^\circ)$ is rational.

Solution: This is TRUE. We use the identities $\sin\alpha\sin\beta = \frac{1}{2}\big(\cos(\alpha-\beta)-\cos(\alpha+\beta)\big)$ and $\sin\alpha\sin\beta = \frac{1}{2}\big(\sin(\alpha+\beta)+\sin(\alpha-\beta)\big)$ and the fact that $\sin(100^\circ)=\cos(10^\circ)=\sin(80^\circ)$ to get

$$\begin{aligned}\sin(20^\circ)\sin(40^\circ)\sin(60^\circ)\sin(80^\circ) &= \tfrac{1}{2}\big(\cos(20^\circ)-\cos(60^\circ)\big)\tfrac{\sqrt{3}}{2}\sin(80^\circ)\\ &= \tfrac{\sqrt{3}}{4}\cos(20^\circ)\sin(80^\circ)-\tfrac{\sqrt{3}}{8}\sin(80^\circ)\\ &= \tfrac{\sqrt{3}}{8}\big(\sin(100^\circ)+\sin(60^\circ)\big)-\tfrac{\sqrt{3}}{8}\sin(80^\circ)\\ &= \tfrac{\sqrt{3}}{8}\sin(100^\circ)+\tfrac{3}{16}-\tfrac{\sqrt{3}}{8}\sin(80^\circ)\\ &= \tfrac{3}{16}\,.\end{aligned}$$

6: $\left(\dfrac{e}{2}\right)^{\sqrt{3}} < \big(\sqrt{2}\big)^{\pi/2}$.

Solution: This is TRUE. We have $\left(\dfrac{e}{2}\right)^{\sqrt{3}} < \big(\sqrt{2}\big)^{\pi/2} \iff \sqrt{3}\ln\left(\frac{e}{2}\right) < \frac{\pi}{2}\ln\sqrt{2} \iff \sqrt{3}(1-\ln 2) < \frac{\pi}{4}\ln 2 \iff \sqrt{3} < \left(\frac{\pi}{4}+\sqrt{3}\right)\ln 2$. We have $\pi > 3.141$ so $\frac{\pi}{4} > .785$ and $\sqrt{3} > 1.732$, and we have $\ln 2 > .69$ and so $\sqrt{3}\left(\frac{\pi}{4}+\sqrt{3}\right)\ln 2 > (2.517)(.69) > 1.736 > \sqrt{3}$.

7: Given $a\in\mathbf{R}$, let $x_1 = a$ and for $n\ge 1$ let $x_{n+1} = x_n\cos(x_n)$. Then $\{x_n\}$ converges for all choices of $a\in\mathbf{R}$.

Solution: This is FALSE. When $a=\pi$ we have $x_n = (-1)^{n+1}\pi$.

8: Define a bijection $f:\mathbf{Z}^+\to\mathbf{Z}^2$ by counting the elements in $\mathbf{Z}^2$ as follows. Let $f(1)=(0,0)$ and $f(2)=(1,0)$, and then continue counting by spiralling counterclockwise so that for example we have

k	1	2	3	4	5	6	7	8	9	10
$f(k)$	$(0,0)$	$(1,0)$	$(1,-1)$	$(0,-1)$	$(-1,-1)$	$(-1,0)$	$(-1,1)$	$(0,1)$	$(1,1)$	$(2,1)$

Then there exists $a\in\mathbf{Z}^+$ such that $f^{-1}(a,0)$ is a multiple of 5.

Solution: This is FALSE. We have $f^{-1}(1,0)=2$, $f^{-1}(2,0)=1+(1+1+2+2)+3+1=11$, $f^{-1}(3,0)=1+(1+1+2+2+3+3+4+4)+5+2=28$, and in general we have

$$\begin{aligned}f^{-1}(a,0) &= 1+2\big(1+2+\cdots+(2a-2)\big)+(2a-1)+(a-1)\\ &= 1+(2a-1)(2a-2)+3a-2 = 4a^2-3a+1\end{aligned}.$$

In $\mathbf{Z}_5$ we have

a	0	1	2	3	4
a^2	0	1	4	4	1
$4a^2-3a+1$	1	2	1	3	3

so we see that $f^{-1}(a,0)$ is never a multiple of 5.

9: There exists a permutation $\{a_1, a_2, \cdots, a_{20}\}$ of the set $\{1, 2, \cdots, 20\}$ such that for all k with $1 < k < 20$, either $a_k = a_{k+1} + a_{k-1}$ or $a_k = |a_{k+1} - a_{k-1}|$.

Solution: This is FALSE. Reduce modulo 2 to get $a_k \in \mathbf{Z}_2$. Then for $1 < k < 20$ we have $a_k = a_{k-1} + a_{k+1}$, that is $a_{k+1} = a_k + a_{k-1}$. Thus modulo 2, the entire sequence $a_1, a_2, \cdots, a_{20}$ is entirely determined from the values a_1, a_2. The only possibilities are $000000000\cdots$, $011011011\cdots$, $101101101\cdots$ and $110110110\cdots$. None of these possibilities can occur since the set $\{1, 2, \cdots, 20\}$ has 10 even numbers and 10 odd numbers.

10: There exists a permutation $\{a_1, a_2, \cdots, a_{20}\}$ of the set $\{1, 2, \cdots, 20\}$ such that for all k with $1 \le k \le 20$, $k + a_k$ is a power of 2.

Solution: This is TRUE. We can use the permutation

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a_k	3	2	1	4	11	10	9	8	7	6	5	20	19	18	17	16	15	14	13	12

11: There exists a partition of $\{1, 2, \cdots, 15\}$ into 5 disjoint 3-element sets $S_k = \{a_k, b_k, c_k\}$ such that $a_k + b_k = c_k$ for $k = 1, 2, 3, 4, 5$.

Solution: This is TRUE. For example, we can use $\{1, 14, 15\}$, $\{2, 7, 9\}$, $\{3, 10, 13\}$, $\{4, 8, 12\}$, $\{5, 6, 11\}$.

12: For every finite set of integers S, $\left|\left\{(a,b) \in S^2 \middle| a-b \text{ is odd}\right\}\right| \le \left|\left\{(a,b) \in S^2 \middle| a-b \text{ is even}\right\}\right|$.

Solution: This is TRUE. Say $S = \{a_1, a_2, \cdots, a_k\} \cup \{b_1, b_2, \cdots, b_l\}$ where the a_i and b_i are distinct with each a_i even and each b_i odd. Then we have

$$\left|\left\{(a,b) \in S^2 \middle| a-b \text{ is odd}\right\}\right| = \left|\left\{(a_i, b_j)\right\}\right| + \left|\left\{(b_i, a_j)\right\}\right| = 2kl \text{ , and}$$
$$\left|\left\{(a,b) \in S^2 \middle| a-b \text{ is even}\right\}\right| = \left|\left\{(a_i, a_j)\right\}\right| + \left|\left\{(b_i, b_j)\right\}\right| = k^2 + l^2$$

and we have $(k^2 + l^2) - 2kl = (k - l)^2 \ge 0$.

13: For every set S, whose elements are finite subsets of $\mathbf{Z}$, with the property that $A \cap B \neq \emptyset$ for all $A, B \in S$, there exists a finite set $C \subset \mathbf{Z}$ such that $A \cap B \cap C \neq \emptyset$ for all $A, B \in S$.

Solution: This is FALSE. For example we can take

$$S = \Big\{\{1,2\}, \{2,3\}, \{1,3,4\}, \{2,4,5\}, \{1,3,5,6\}, \{2,4,6,7\}, \{1,3,5,7,8\}, \{2,4,6,8,9\}, \cdots\Big\}$$

14: There exists a linearly independent set $\{A_1, A_2, A_3\}$ of real 3×3 matrices such that every non-zero matrix in $\mathrm{Span}\{A_1, A_2, A_3\}$ is invertible.

Solution: This is FALSE. Indeed we can not even find two such matrices. Let $\{A, B\}$ be a linearly independent set of 3×3 real matrices. Since $\det(A + tB)$ is a real polynomial of degree 3 in t, it has a real root, so we can choose $t \in \mathbf{R}$ so that $A + tB$ is not invertible.

15: For all 2×2 real matrices A, B and C, $\det\begin{pmatrix} I & A \\ B & C\end{pmatrix}=0$ if and only if $\det\begin{pmatrix} I & B \\ A & C\end{pmatrix}=0$.

Solution: This is FALSE. For example, take $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $B=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $C=\begin{pmatrix}1&0\\0&0\end{pmatrix}$.

16: There exists a positive integer n and an $n\times n$ matrix A whose entries lie in $\{0,1\}$, such that $\det(A)>n$.

Solution: This is TRUE. For example, let $A=\begin{pmatrix}B&0&0&0\\0&B&0&0\\0&0&B&0\\0&0&0&B\end{pmatrix}$ where $B=\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}$.

Then A is a 12×12 matrix and $\det(A)=\big(\det B\big)^4=2^4=16$.

17: For every function $f:\mathbf{R}\to\mathbf{R}$, if f^2 and f^3 are both polynomials, then so is f.

Solution: This is TRUE. Say $f^2=g$ and $f^3=h$ where g and h are polynomials. Write $g=a\,{p_1}^{k_1}{p_2}^{k_2}\cdots{p_m}^{k_m}$ and $h=b\,{p_1}^{l_1}{p_2}^{l_2}\cdots{p_m}^{l_m}$ where $a,b\in\mathbf{R}$, the p_i are distinct irreducible polynomials, and each $k_i,l_i\geq 0$. Since $g^3=h^2$ we have $3k_i=2l_i$ for all i. Thus each l_i is a multiple of 3, say $l_i=3t_i$. Since $f^3=h=b\,{p_1}^{3t_1}{p_2}^{3t_2}\cdots{p_m}^{3t_m}$, we have $f=\sqrt[3]{b}\,{p_1}^{t_1}{p_2}^{t_2}\cdots{p_m}^{t_m}$, which is a polynomial.

18: Every real polynomial is equal to the difference of two increasing polynomials.

Solution: This is TRUE. Choose an even inter $m>\deg(f')$. Since $\lim\limits_{x\to\pm\infty}(x^m+f')=\infty$, it follows that x^m+f' has an absolute minimum. Chose $a>0$ so that $x^m+f'(x)+a>0$ for all $x\in\mathbf{R}$. Let $g(x)=\frac{1}{m+1}\,x^{m+1}+ax$. Since $g'(x)=x^m+a\geq a>0$ for all $x\in\mathbf{R}$, g is increasing. Since $(f+g)'(x)=x^m+f'(x)+a>0$ for all $x\in\mathbf{R}$, $f+g$ is increasing. Thus $f=(f+g)-g$ is the difference of two increasing polynomials.

19: For every polynomial f with integer coefficients, and for all distinct integers $a_1,a_2,\cdots,a_l$, there exists an integer c such that the product $p(a_1)p(a_2)\cdots p(a_l)$ divides $f(c)$.

Solution: This is FALSE. For example, let $f(x)=2x^2+2$. Then $f(0)=2$ and $f(1)=4$ but there is no integer c such that $f(c)$ is a multiple of 8. Indeed, for $x\in\mathbf{Z}_8$ we have $x^2\in\{0,1,4\}$ so $f(x)\in\{2,4\}$.

20: For all increasing functions $f,g:\mathbf{R}\to\mathbf{R}$ with $f(x)<g(x)$ for all $x\in\mathbf{Q}$, we have $f(x)\leq g(x)$ for all $x\in\mathbf{R}$.

Solution: This is FALSE. For example, we can take $f(x)=\begin{cases}2(x-\sqrt{2})\ , & \text{for } x<\sqrt{2}\\ x-\sqrt{2}+1\ , & \text{for } x\geq\sqrt{2}\end{cases}$

and $g(x)=\begin{cases}x-\sqrt{2}\ , & \text{for } x\leq\sqrt{2}\\ 2(x-\sqrt{2})+1\ , & \text{for } x>\sqrt{2}\,.\end{cases}$

21: There exists a continuously differentiable function $f : \mathbf{R} \to \mathbf{R}^+$ such that $f'(x) = f\big(f(x)\big)$ for all $x \in \mathbf{R}$.

Solution: This is FALSE. Suppose, for a contradiction, that f is such a function. Since $f'(x) = f(f(x)) \in \mathbf{R}^+$ for all $x \in \mathbf{R}$, f is increasing. For all $x \in \mathbf{R}$ we have $f(x) \in \mathbf{R}^+$, that is $f(x) > 0$, so since f is increasing $f(f(x)) > f(0)$, and so $f'(x) = f(f(x)) > f(0)$. It follows, from the Mean Value Theorem, that all $x < 0$ we have $f(x) < f(0) + f'(0)\,x$ $\big($indeed, we have $\frac{f(x)-f(0)}{x-0} = f'(c) > f(0)$ for some $x < c < 0$, and so, since $x < 0$, we have $f(x) - f(0) < x\,f(0)$, that is $f(x) < f(0) + f'(0)\,x\big)$. In particular, when $x = -\frac{f(0)}{f'(0)}$ we have $f(x) < f(0) - f'(0) \cdot \frac{f(0)}{f'(0)} = 0$, which is not possible since f takes values in $\mathbf{R}^+$.