Solutions to the Bernoulli Trials Problems for 2011

1: There exists a positive integer n such that for every integer a with $1{,}000 \le a \le 1{,}000{,}000$, a is prime if and only if $\gcd(a,n) = 1$.

Solution: This is TRUE. Indeed we can take n to be the product of all primes p with $p < 1{,}000$.

2: For all irrational numbers x and y such that y is not a rational multiple of x, the set $\big\{\big(\langle tx\rangle, \langle ty\rangle\big)\big| t \in \mathbf{Z}\big\}$ is dense in the unit square $[0,1]\times[0,1]$. (Here $\langle x\rangle$ denotes the *fractional part* of x, that is $\langle x\rangle = x - \lfloor x\rfloor$).

Solution: This is FALSE. For example, if we take $x = \sqrt{2}$ and $y = 1+\sqrt{2}$, then $\langle tx\rangle = \langle ty\rangle$ for all $t \in \mathbf{Z}$, and so each point $\big(\langle tx\rangle, \langle ty\rangle\big)$ lies on the main diagonal in the square $[0,1]\times[0,1]$, and the diagonal is not dense.

3: For every positive integer n, the number of ordered pairs of positive integers (a,b) with $\operatorname{lcm}(a,b) = n$ is equal to the number of positive divisors of n^2.

Solution: This is TRUE. Let $n = {p_1}^{k_1}{p_2}^{k_2}\cdots{p_l}^{k_l}$ be the prime decomposition of n. We have $\operatorname{lcm}(a,b) = n$ when $a = {p_1}^{i_1}{p_2}^{i_2}\cdots{p_l}^{i_l}$ and $b = {p_1}^{j_1}{p_2}^{j_2}\cdots{p_l}^{k_l}$ with $\max\{i_\alpha, j_\alpha\} = k_\alpha$ for each $\alpha = 1,2,\cdots,l$. For each α, there are $2k_\alpha + 1$ choices for (i_α, j_α), namely $(i_\alpha, j_\alpha)\big\{(0,k),(1,k),\cdots,(k,k),(k,k-1),\cdots,(k,1),(k,0)\big\}$. Thus the total number of such pairs (a,b) is $\prod\limits_{\alpha=1}^{l}(2k_\alpha+1)$, which is the number of positive divisors of n^2.

4: The last non-zero digit of 100! is equal to 4.

Solution: This is TRUE. The number of zeros at the end of 100! is equal to the number of factors of 5 in 100!, which is equal to $20+4 = 14$. To find the last non-zero digit we calculate $\frac{100!}{10^{14}}$ (mod 10). We note that $\frac{100!}{10^{14}} = 0$ (mod 2) since 100! contains more than 14 factors of 2, so it suffices to find $\frac{100!}{10^{14}}$ (mod 5). We have

$$\begin{aligned}
\frac{100!}{10^{14}} &= \frac{1}{2^{14}}\cdot\frac{100!}{5^{14}} \\
&= \frac{1}{2^{14}}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{1}\cdot 6\cdot 7\cdot 8\cdot 9\cdot \mathbf{2}\cdot 11\cdot 12\cdot 13\cdot 14\cdot \mathbf{3}\cdot 16\cdot 17\cdot 18\cdot 19\cdot \mathbf{4} \\
&\qquad\cdot 21\cdot 22\cdot 23\cdot 24\cdot \mathbf{1}\cdots\cdot \mathbf{18}\cdot 91\cdot 92\cdot 93\cdot 94\cdot \mathbf{19}\cdot 96\cdot 97\cdot 98\cdot 99\cdot \mathbf{4}
\end{aligned}$$

Working modulo 5 we have $2^{-1} = 3$ and so

$$\begin{aligned}
\frac{100!}{10^{14}} = 3^{14} &\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{1}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{2}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{3}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{4}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{1} \\
&\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{1}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{2}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{3}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{4}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{2} \\
&\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{1}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{2}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{3}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{4}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{3} \\
&\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{1}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{2}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{3}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{4}\cdot 1\cdot 2\cdot 3\cdot 4\cdot \mathbf{4} \\
= 3^{14}&\cdot(1\cdot 2\cdot 3\cdot 4)^{20+4+1} = 3^{14}\cdot 4^{25} = 3^0\cdot 4^1 = 4 \pmod 5,
\end{aligned}$$

where on the last line, we used Fermat's Little Theorem to obtain $3^{14} = 3^0$ and $4^{25} = 4^1$. Finally note that since $\frac{100!}{10^{14}} = 0$ (mod 2) and $\frac{100!}{10^{14}} = 4$ (mod 5), we have $\frac{100!}{10^{14}} = 4$ (mod 10).

5: For every integer $n>1$, n is prime if and only if $\sin\left(\dfrac{1+(n-1)!}{n}\,\pi\right)=0$.

Solution: This is TRUE. Indeed we have

$$\sin\left(\frac{1+(n-1)!}{n}\,\pi\right)=0 \iff \frac{1+(n-1)!}{n}\in\mathbf{Z} \iff (n-1)!=-1 \pmod{n}\,,$$

and Wilson's Theorem states that for every integer $n>1$, n is prime if and only if $(n-1)!=-1 \pmod{n}$.

For completeness, we provide a proof of Wilson's Theorem. First suppose that n is prime. Consider the polynomial $f(x)=x^{n-1}-1$. By Fermat's Little Theorem, every non-zero element of $\mathbf{Z}_n$ is a root of f, so we must have $f(x)=(x-1)(x-2)\cdots(x-(n-1))$. Setting $x=0$ gives $-1=f(0)=(-1)(-2)\cdots(-(n-1))=(n-1)(n-2)\cdots(1)=(n-1)!$ in $\mathbf{Z}_n$. Conversely suppose that n is composite, say $n=ab$ where $1<a\leq b<n$. In the case that $a<b$ we have $(n-1)!=1\cdot 2\cdots a\cdots b\cdot(n-1)$ so $n=ab\,\big|\,(n-1)!$, that is $(n-1)!=0 \pmod{n}$. In that case that $2<a=b$ we have $(n-1)!=1\cdot 2\cdots a\cdots 2a\cdots(n-1)$ so $n=a^2\,\big|\,(n-1)!$ and again $(n-1)!=0 \pmod{n}$. In the case that $2=a=b$ and we have $n=4$ so $(n-1)!=2 \pmod{n}$.

6: Every periodic function $f:\mathbf{R}\to\mathbf{R}$ has a unique smallest positive period. (A function $f:\mathbf{R}\to\mathbf{R}$ is called *periodic* with *period* $p>0$ when $f(x+p)=f(x)$ for all $x\in\mathbf{R}$).

Solution: This is FALSE. For example, every positive rational number is a period for the function

$$f(x)=\begin{cases}1 \text{ if } x\in\mathbf{Q}\\ 0 \text{ if } x\notin\mathbf{Q}\end{cases}$$

7: There is a parabola which is tangent to every line whose x and y-intercepts add up to 1.

Solution: This is TRUE. One nice way to find the equation of the parabola is as follows. We construct a map $f:\mathbf{R}^2\to\mathbf{R}^2$ which sends the vertical lines (in say the uv-plane) to the lines (in the xy-plane) whose x and y-intercepts add up to 1, for example we can send the line $(u,v)=(a,0)+t(0,1)=(a,t)$ to the line $(x,y)=(a,0)+t(-a,1-a)=(a-at,t-at)$ by setting $(x,y)=f(u,v)=\big(u-uv,v-uv\big)$. Then we have

$$Df=\begin{pmatrix}x_u & x_v\\ y_u & y_v\end{pmatrix}=\begin{pmatrix}1-v & -u\\ -v & 1-u\end{pmatrix}.$$

The map f fails to be 1:1 when $0=\det(Df)=(1-u)(1-v)-uv=1-u-v$, that is along the line $u+v=1$. Our map f folds the uv-plane along the line $u+v=1$ and then sends it to the desired parabola in the xy-plane. To find the image of the line $u+v=1$ we describe it parametrically by $(u,v)=(1,0)+t(-1,1)=(1-t,t)$, then we see that the map f sends the line to the curve $(x,y)=f(1-t,t)=\big(1-t-(1-t)t,t-(1-t)t\big)=\big(1-2t+t^2,t^2\big)$. To see that this curve is a parabola, we note that $x+y=1-2t+2t^2$ and $x-y=1-2t$ so the curve has equation $2(x+y)=1+(x-y)^2$. We leave it as an exercise to verify that this parabola is indeed tangent to all of the lines whose x and y-intercepts add up to 1.

8: Let $a_1=a_2=a_3=1$ and let $a_n=\dfrac{1+a_{n-1}a_{n-2}}{a_{n-3}}$ for $n\geq 4$. Then each a_n is an integer.

Solution: This is TRUE. The first few terms in the sequence are $1,1,1,2,3,7,11,26,41,97$. The sequence of odd terms begins with $1,1,3,11,41$ and the sequence of even terms begins with $1,2,7,26,97$. It appears that $\{a_n\}$ satisfies the recurrence $a_n=4a_{n-2}-a_{n-4}$. To show this, let $\{b_n\}$ be the sequence given by $b_1=b_2=b_3=1$ and $b_4=2$, with $b_n=4b_{n-2}-b_{n-4}$ for $n\geq 5$. Writing $c_n=b_nb_{n-3}-b_{n-1}b_{n-2}$ for $n\geq 4$, we have

$$c_n=b_nb_{n-3}-b_{n-1}b_{n-2}=(4b_{n-2}-b_{n-4})b_{n-3}-(4b_{n-3}-b_{n-5})b_{n-2}=b_{n-2}b_{n-5}-b_{n-3}b_{n-4}=c_{n-2}$$

for all $n\geq 6$, and since $c_5=c_4=1$ we have $c_n=1$ for all $n\geq 4$. Thus $b_nb_{n-3}-b_{n-1}b_{n-3}=1$ for all $n\geq 4$, that is $b_n=\frac{1+b_{n-1}b_{n-2}}{b_{n-3}}$ for all $n\geq 4$, so $\{b_n\}$ satisfies the same recurrence equation as $\{a_n\}$ and we have $a_n=b_n$ for all $n\geq 1$. Thus $\{a_n\}$ satisfies the recurrence $a_n=4a_{n-2}-a_n$ as claimed, so each $a_n\in\mathbf{Z}$.

9: At each point $(a,b) \in \mathbf{Z}^2 \setminus \{(0,0)\}$, there is a cylinder of height 1 whose base is a circle of radius $\frac{3}{10}$ centered at (a,b). Exactly 24 of these cylinders can be seen from the origin.

Solution: This is TRUE. With the help of a picture drawn approximately to scale, it is easy to see that the cylinders centered at $(\pm 1,0),(0,\pm 1),(\pm 1,\pm 1),(\pm 1,\pm 2)$ and $(\pm 2,\pm 1)$ are visible from $(0,0)$ and that the only other cylinders which might be visible are centered at $(\pm 1,\pm 3),(\pm 3,\pm 1),(\pm 2,\pm 3)$ and $(\pm 3,\pm 2)$.

To determine whether the cylinder centered at $(3,1)$ is visible, we find the values of m such that the line L_m with equation $y = mx$ is equidistant from the points $(1,0)$ and $(2,1)$. Let $d_1 = \text{dist}\big((1,0),L_m\big) = \frac{|m|}{\sqrt{1+m^2}}$ and $d_2 = \text{dist}\big((2,1),L_m\big) = \frac{|2m-1|}{\sqrt{1+m^2}}$. To have $d_1 = d_2$ we need $|m| = |2m-1|$, that is $m = 1$ or $\frac{1}{3}$, and when $m = \frac{1}{3}$ we have $d_1 = d_2 = \frac{1/3}{\sqrt{10/9}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10} > \frac{3}{10}$. Thus the line $y = \frac{1}{3}x$ passes between the cylinders centered at $(1,0)$ and $(2,1)$, so the cylinder at $(3,0)$ is visible from the origin.

Similarly, to determine whether the cylinder centered at $(3,2)$ we find the values of m such that the line L_m with equation $y = mx$ is equidistant from $(1,1)$ and $(2,1)$. Let $d_1 = \text{dist}\big((1,1),L_m\big) = \frac{|m-1|}{\sqrt{m^2+1}}$ and $d_2 = \text{dist}\big((2,1),L_m\big) = \frac{|2m-1|}{\sqrt{m^2+1}}$. To have $d_1 = d_2$ we need $|m-1| = |2m-1|$, that is $m = 1$ or $\frac{2}{3}$, and when $m = \frac{2}{3}$ we have $d_1 = d_2 = \frac{1/3}{\sqrt{\frac{13}{9}}} = \frac{1}{\sqrt{13}} = \frac{3}{\sqrt{117}} < \frac{3}{10}$. Thus no line $y = mx$ passes between the cylinders at $(1,1)$ and $(2,1)$, so the cylinder at $(3,2)$ is not visible from the origin.

By symmetry, we see that there are exactly 24 cylinders visible from the origin, namely those centered at $(\pm 1,0),(0,\pm 1),(\pm 1,\pm 1),(\pm 1,\pm 2),(\pm 2,\pm 1),(\pm 3,\pm 1)$ and $(\pm 1,\pm 3)$.

10: Let S be the unit circle $x^2 + y^2 = 1$ and let T be the unit circle with the point $(1,0)$ removed. Then T can be partitioned into two disjoint non-empty sets A and B such that for some rotation R about the origin, the sets A and $R(B)$ form a partition of S.

Solution: This is TRUE. Let $\alpha = e^{i\theta}$ where θ is an irrational multiple of π (for example, we could take $\theta = 1$). Let $B = \{\alpha, \alpha^2, \alpha^3, \cdots\}$ and let $A = T \setminus B$. Let R be the rotation about the origin by the angle $-\theta$. Then $R(B) = \{1, \alpha, \alpha^2, \cdots\}$ and we see that A and $R(B)$ form a partition of S.

11: The entries of an $n \times n$ matrix A are chosen at random from $\{1,2,3,\cdots,100\}$. Let P_n be the probability that $\det(A)$ is odd. Then $0 < \lim\limits_{n\to\infty} P_n < \frac{1}{2}$.

Solution: This is TRUE. Consider A as an $n \times n$ matrix with entries in $\mathbf{Z}_2$ so that $\det(A)$ is odd if and only if A is invertible. In order for A to be invertible, the first row must be non-zero so there are $2^n - 1$ choices for the first row, then the second row cannot be a multiple of the first row (there are 2 such multiples) so there are $2^n - 2$ choices for the second row, then the third row cannot be a linear combination of the first two rows (there are $2^2 = 4$ such linear combinations) so there are $2^n - 4$ choices for the third row, and so on. The total number of invertible $n \times n$ matrices is $(2^n-1)(2^n-2)(2^n-4)\cdots(2^n-2^{n-1})$. Since the total number of $n \times n$ matrices is $(2^{n^2}) = (2^n)^n$, the probability that A is invertible is $P_n = \left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{2^{n-1}}\right)\cdots\left(1 - \frac{1}{2}\right)$. Note that $\{P_n\}$ is positive and decreasing so it converges. For $0 < x < \frac{1}{2}$ we have $-2x < \ln(1-x) < -x$, and so it follows that $\ln P_n < -\frac{1}{2^n} - \frac{1}{2^{n-1}} - \cdots - \frac{1}{2} \to -1$ so $\lim\limits_{n\to\infty} P_n < e^{-1}$, and that $\ln P_n > -\frac{1}{2^{n+1}} - \frac{1}{2^n} - \cdots - \frac{1}{4} \to -\frac{1}{2}$ so $\lim\limits_{n\to\infty} P_n > e^{-2}$. Thus $e^{-2} < \lim\limits_{n\to\infty} P_n < e^{-1}$.

12: For every function $f : [0,1] \to [0,1]$ which is continuous and non-decreasing, the length of the graph of f is less than 2. (The *length* of the graph of f is the supremum, over all partitions $0 = x_0 < x_1 < \cdots < x_n = 1$, of the sum $\sum\limits_{i=1}^{n} \sqrt{(\Delta_i x)^2 + (\Delta_i y)^2}$ where $\Delta_i x = x_i - x_{i-1}$ and $\Delta_i y = f(x_i) - f(x_{i-1})$).

Solution: This is FALSE. Indeed the graph of the Cantor function has length exactly equal to 2. For completeness, we include a brief description of the Cantor set and the Cantor function.

The **Cantor set** $C \subset [0,1]$ is an uncountable set of measure zero constructed as follows. Begin with the set $C_0 = [0,1]$. Delete the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ to obtain the set $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Delete the open middle thirds of these two closed intervals to obtain $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. Continue this process to obtain closed sets $C_0 \supset C_1 \supset C_2 \supset \cdots$ where C_k is a disjoint union of 2^k closed intervals each of size $\frac{1}{3^k}$. The Cantor set is the set $C = \bigcap\limits_{k=1}^{\infty} C_k \subset [0,1]$. As an exercise, show that C is the set of numbers which can be written in base 3 as $0.a_1 a_2 a_3 \cdots$ with each $a_i \in \{0, 2\}$.

The **Cantor function** $f : [0,1] \to [0,1]$ is defined as follows. For $x \in C$, we write x in base 3 as $x = 0.a_1 a_2 a_3 \cdots$ with each $a_i \in \{0, 2\}$, then we define $f(x)$ to be the base 2 number $f(x) = 0.\frac{a_1}{2} \frac{a_2}{2} \frac{a_3}{2} \cdots$. As an exercise, show that for $x, y \in C$ with $x < y$ we have $f(x) = f(y)$ if and only if x and y are of the form $x = 0.a_1 \cdots a_k 0222 \cdots$ and $y = 0.a_1 \cdots a_k 2000 \cdots$, that is when the open interval (x, y) is one of the open middle thirds deleted in the construction of C. We also leave it as an exercise to verify that f is continuous and non-decreasing.

Verify that when use the partition $0 = x_0 < x_1 < \cdots < x_n = 1$, where the x_i are the 2^{k+1} endpoints of the 2^k closed intervals in C_k, we obtain $\sum\limits_{i=1}^{n} \sqrt{(\Delta_i x)^2 + (\Delta_i y)^2} \geq 1 + \left(\frac{1}{3} + \frac{2}{9} + \cdots + \frac{2^{k-1}}{3^k}\right) = 2 - \left(\frac{2}{3}\right)^k \to 2$ as $k \to \infty$.

13: A permutation σ of $\{1, 2, 3, \cdots, 200\}$ is chosen at random. The probability that σ contains a cycle of length exactly 100 is less than 1%.

Solution: This is TRUE. We find the number of permutations consisting of two cycles of length 100. In cycle notation, such a permutation is of the form $(a_1 a_2 \cdots a_{100})(b_1 b_2 \cdots b_{100})$, where we can take $a_1 = 1$ then there are 199 choices for a_2, 198 choices for a_3 and so on until we reach 101 choices for a_{100}, and then we can take b_1 to be the smallest remaining element of $\{1, 2, \cdots, 200\}$ not equal to any a_i, and then there are 99 choices for b_2, 98 choices for b_3 and so on. Thus the number of permutations consisting of two cycles of length 100 is equal to $\frac{200!}{200 \cdot 100}$. Next we find the number of permutations which contain exactly one cycle of length 100. Such a permutation is of the form $(a_1 a_2 \cdots a_n)\,\beta$ where β is not a cycle of length 100. There are $\binom{200}{100}$ ways to choose the numbers a_i, then we can take a_1 to be the smallest of these, and then there are 99 choices for a_2, 98 choices for a_3 and so on. Then β is a permutation of the 100 remaining elements in $\{1, 2, \cdots, 200\}$ not equal to any a_i. There are 100! such permutations, but 99! of these are cycles of length 100, so this gives $(100! - 99!) = 99! \cdot 99$ choices for β. Thus the number of permutations which contain exactly one cycle of length 100 is equal to $\binom{200}{100} \cdot 99! \cdot 99! \cdot 99$. Thus the number of permutations σ which contain (at least) one cycle of length 100 is equal to $\frac{200!}{200 \cdot 100} + \frac{200!}{100! \cdot 100!} \cdot 99! \cdot 99! \cdot 99 = \frac{200!}{100 \cdot 100}\left(\frac{1}{2} + 99\right)$. Since the total number of permutations is equal to 200!, the desired probability is $P = \frac{99.5}{10000} = 0.995\%$.

14: For every positive integer n there exists a binary string $s = a_1 a_2 \cdots a_l$ of length $l = 2^n + n - 1$ such that each of the 2^n binary strings of length n occurs as a substring of s.

Solution: This is TRUE. For those who know some graph theory, we can see that such a binary string s exists as follows. We construct a directed graph G whose vertices are the 2^{n-1} binary stings of length $n-1$. From the vertex $a_1a_2a_3\cdots a_{n-1}$ we have one directed edge (labeled by 1) leading to the vertex $a_2a_3\cdots a_{n-1}1$ and we have one directed edge (labeled by 0) leading to the vertex $a_2a_3\cdots a_{n-1}0$. This graph G is strongly connected (meaning that there is a directed path from any given vertex to any other given vertex) and each vertex has exactly two edges leading from it and two edges leading to it. It follows that G admits an Eulerian circuit, From an Eulerian circuit we can construct a sting s as desired by letting s be given by the initial vertex of the path followed by the labels of the edges of the path.

For those who do not know any graph theory, it is still possible to construct such a string s in various ways. Here is one construction. Let the initial n-string of s be $a_1a_2\cdots a_n = 00\cdots 0$, then for each $k \geq n$, having constructed $s_k = a_1a_2\cdots a_k$, choose a_{k+1} as follows. If the n-string $a_{k-n+2}\cdots a_{k-1}a_k1$ has not yet occurred in s_k then choose $a_{k+1} = 1$; otherwise, if the n-string $a_{k-n+2}\cdots a_{k-1}a_k0$ has not yet occurred in s_k then choose $a_{k+1} = 0$; otherwise both of the strings $a_{k-n+2}\cdots a_{k-1}a_k1$ and $a_{k-n+2}\cdots a_{k-1}a_k0$ have already occurred in s_k, and in this case we halt having completed the string $s = s_k$. We shall show that this construction produces a string s of the required form.

First we claim that our sequence $s = a_1a_2\cdots a_l$ must end with the n-string $a_{l-n+1}\cdots a_{l-1}a_l = 10\cdots 00$. Since our procedure ends at step l, it follows that the strings $a_{l-n+2}\cdots a_{l-1}a_l1$ and $a_{l-n+2}\cdots a_{l-1}a_l0$ have both occured previously in s, and so the final $(n-1)$-string $a_{l-n+2}\cdots a_{l-1}a_l$ has just occurred for the third time. The final two of these three occurrences gave us the two n-strings $1a_{l-n+2}\cdots a_{l-1}a_l$ and $0a_{l-n+2}\cdots a_{l-1}a_l$, and so the first of the three occurrences must have occurred as the initial $(n-1)$-string of s, that is we must have $a_{l-n+2}\cdots a_{l-1}a_l = 0\cdots 00$. Since the n-string $00\cdots 0$ occurred at the beginning of s, the final n-string must be $10\cdots 0$, as claimed.

Now we claim that every n-string occurs in s. We know that the n-strings $00\cdots 0$ and $10\cdots 0$ both occur in s. Suppose, inductively, that all 2^k of the n-strings of the form $a_1\cdots a_k0\cdots 0$ occur in s. Since the n-string $a_1\cdots a_k0\cdots 00$ occurs, the n-string $a_1\cdots a_k0\cdots 01$ must have occurred earlier. Thus the $(n-1)$-string $a_1\cdots a_k0\cdots 0$ occurs twice. Hence the n-strings $1a_1\cdots a_k0\cdots 0$ and $0a_1\cdots a_k0\cdots 0$ both occur. Thus all 2^{k+1} of the n-strings of the form $a_0a_1a_2\cdots a_k0\cdots 0$ occur in s. By induction, every n-string occurs in s.

15: Let $a_1 = 2$ and for $n \geq 1$ let $a_{n+1} = \dfrac{a_n(n+a_n)}{n+1}$. Then each a_n is an integer.

Solution: This is FALSE. To show this, it suffices to find a value of n such that $(n+1)$ does not divide $a_n(n+a_n)$. To find such a value of n we consider the case that $p = n+1$ is prime and we work modulo p. Since the first 6 terms are $2, 3, 5, 10, 28, 154$ we begin with the case $p = 11$. Working modulo 11 gives

k	1	2	3	4	5	6	7	8	9	10
a_k	2	3	5	10	6	0	0	0	0	0

so again we find no contradiction. Working modulo 13 gives

k	1	2	3	4	5	6	7	8	9	10	11	12
a_k	2	3	5	10	2	11	10	5	0	0	0	0

so again we find no contradiction. Working modulo 17 gives

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_k	2	3	5	10	11	6	1	1	1	1	1	1	1	1	1	1

so yet again we find no contradiction. Not discouraged, we work modulo 19, 23, 29, 31, 37, 41 to no avail, then finally working modulo 43 gives

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
a_k	2	3	5	10	28	25	37	10	20	15	38	19	42	36	34	2	35	39	31	13	2

22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
6	26	28	29	4	14	42	5	20	17	4	20	16	29	42	13	42	20	8	23	33

and lo and behold we find that 43 does not divide $a_{42}(42 + a_{42})$, so a_{43} is not an integer.