Solutions to the Bernoulli Trials Problems for 2010

1: There exists a rational number x with $x \neq \pm 1$ such that $x + \frac{1}{x}$ is an integer.

Solution: This is FALSE. Suppose that $x \in \mathbf{Q}$ with $x+\frac{1}{x} = n \in \mathbf{Z}$. Then $x^2 - nx + n = 0$ and so $x = \frac{n \pm \sqrt{n^2-4}}{2}$. Since $x \in \mathbf{Q}$ we must have $\sqrt{n^2-4} \in \mathbf{Q}$ and so $n^2 - 4$ must be a perfect square. This implies that $n = \pm 2$ because when $n \geq 3$ we have $n^2 - (n-1)^2 = 2n - 1 \geq 5$. Since $n = \pm 2$ we have $x = \frac{n \pm \sqrt{n^2-4}}{2} = \frac{\pm 2}{2} = \pm 1$.

2: For every positive integer n there exist positive integers x, y and z such that $x^2 + y^2 = z^n$.

Solution: This is TRUE. Indeed we have $(2^k)^2 + (2^k)^2 = 2^{2k+1}$ and $(3 \cdot 5^{k-1})^2 + (4 \cdot 5^{k-1})^2 = 5^{2k}$.

3: If there exists a triangle with sides of lengths a, b and c, then there also exists a triangle with sides of lengths $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$.

Solution: This is TRUE. Suppose there exists a triangle with sides of lengths a, b and c with say $a \leq b \leq c$. Note that this implies that $c < a + b$. To show that there is a triangle with sides of lengths $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$, it suffices to show that $\sqrt{c} < \sqrt{a} + \sqrt{b}$, and indeed we have $c < a + b < a + 2\sqrt{ab} + b = \left(\sqrt{a} + \sqrt{b}\right)^2$ and so $\sqrt{c} < \sqrt{a} + \sqrt{b}$, as required.

4: Let p_n be the probability that a number selected at random from the set $\{1, 2, 3, \cdots, n\}$ has its leading digit equal to 1. Then $\lim\limits_{n\to\infty} p_n = \frac{1}{9}$.

Solution: This is FALSE, indeed we claim that $\lim\limits_{n\to\infty} p_n$ does not exist. Notice that 1 of the 9 numbers in $\{1, 2, \cdots, 9\}$ begins with 1, and 10 of the 90 numbers in $\{10, 11, \cdots, 99\}$ begin with 1, and 100 of the 900 numbers in $\{100, 101, \cdots, 999\}$ begin with 1, and so on. It follows that when $n = 10^k - 1 = 99\cdots 9$ we have $p_n = \frac{1+10+\cdots+10^{k-1}}{9+90+\cdots+9\cdot 10^{k-1}} = \frac{1}{9}$, and so if $\lim\limits_{n\to\infty} p_n$ existed then the limit would be $\frac{1}{9}$. But it also follows that when $n = 2\cdot 10^k - 1 = 199\cdots 9$ we have $p_n = \frac{1+10+\cdots+10^k}{2\cdot 10^k - 1} = \frac{(10^{k+1}-1)/9}{2\cdot 10^k - 1} \longrightarrow \frac{5}{9}$ as $k \to \infty$, and so if $\lim\limits_{n\to\infty} p_n$ existed then the limit would be $\frac{5}{9}$.

5: If the sequence $\{a_n\}$ is bounded with $\lim\limits_{n\to\infty} (a_n - a_{n+1}) = 0$ then it converges.

Solution: This is FALSE. A counterexample is given by the following sequence:

$$0, 1, \tfrac{1}{2}, 0, \tfrac{1}{3}, \tfrac{2}{3}, 1, \tfrac{3}{4}, \tfrac{2}{4}, \tfrac{1}{4}, 0, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, 1, \tfrac{5}{6}, \tfrac{4}{6}, \tfrac{3}{6}, \tfrac{2}{6}, \tfrac{1}{6}, 0, \tfrac{1}{7}, \tfrac{2}{7}, \cdots$$

6: There exists a bounded sequence $\{a_n\}_{n\geq 1}$ of real numbers with the property that for all $k > l \geq 1$ we have $|a_k - a_l| \geq \frac{1}{k-l}$.

Solution: This is TRUE. One such sequence begins with $a_0 = 0$ and $a_1 = 2$ and for $0 \leq k < 2^n$ we have $a_{2^n+k} = a_k + \frac{1}{2^{n-1}}$. The first few terms are

$$0, 2, 1, 3, \tfrac{1}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{7}{2}, \tfrac{1}{4}, \tfrac{9}{4}, \tfrac{5}{4}, \tfrac{13}{4}, \tfrac{3}{4}, \tfrac{11}{4}, \tfrac{7}{4}, \tfrac{15}{4}, \tfrac{1}{8}, \cdots$$

7: There exists a quadratic $f(x) = ax^2 + bx + c$ with integral coefficients whose discriminant is equal to 23.

Solution: This is FALSE. If we had $b^2 - 4ac = 23$ for $a, b, c \in \mathbf{Z}$ then reducing modulo 4 would give $b^2 = 3 \in \mathbf{Z}^4$.

8: There exists a cubic $f(x) = ax^3 + bx^2 + cx + d$ with integral coefficients such that $f(19) = 1$ and $f(62) = 2$.

Solution: This is FALSE. Indeed there is no polynomial $f(x)$ with integer coefficients such that $f(19) = 1$ and $f(62) = 2$. Suppose, for a contradiction, that such a polynomial $f(x)$ exists. Let $g(x) = f(x+19)$. Note that $g(x)$ is also a polynomial with integer coefficients and we have $g(0) = f(19) = 1$ and $g(43) = f(62) = 2$. But this is not possible since the fact that $g(0) = 0$ implies that $x \big| g(x)$ for all integers x, and in particular $g(43)$ must be a multiple of 43.

9: Let $f(x)$ be positive and continuous for $x \in [0,\infty)$. If $\displaystyle\int_0^\infty f(x)\,dx$ converges then so does $\displaystyle\int_0^\infty f(x)^2\,dx$.

Solution: This is FALSE. For a counterexample, we construct a function $f(x)$ whose graph has a sequence of spikes, centred at each positive integer n, which grow taller and narrower as n increases. Let

$$g_0(x) = \begin{cases} 1-|x| & \text{, if } |x| \le 1, \\ 0 & \text{, otherwise.} \end{cases}$$

For $n \in \mathbf{Z}^+$, let $g_n(x) = 2^n g_0(4^n x)$. Verify that $\int_{-\infty}^\infty g_n(x)\,dx = \frac{1}{2^n}$ and that $\int_{-\infty}^\infty g_n(x)^2\,dx = \frac{2}{3}$. We let $f(x) = \sum\limits_{n=1}^\infty g_n(x-n)$ for $x \in [0,\infty)$, and then $\int_0^\infty f(x)\,dx = \sum\limits_{n=1}^\infty \frac{1}{2^n} = 1$ but $\int_0^\infty f(x)^2\,dx = \infty$.

10: There exist 100 (not necessarily distinct) positive integers whose sum is equal to their least common multiple.

Solution: This is TRUE. For example, if we let $a_1 = a_2 = \cdots = a_{84} = 1$ and $a_{85} = a_{86} = \cdots a_{98} = 2$ and $a_{99} = 3$ and $a_{100} = 23$ then $\sum\limits_{i=1}^{100} a_i = 84\cdot 1 + 14\cdot 2 + 3 + 23 = 138$ and $\mathrm{lcm}(a_1,\cdots,a_{100}) = 2\cdot 3\cdot 23 = 138$.

11: There exist 7 distinct primes, all less than 900, which are in arithmetic progression.

Solution: This is FALSE. Suppose that $a, a+d, a+2d, \cdots, a+6d$ are all prime, where $a, d \in \mathbf{Z}^+$. Since a is prime we have $a \ge 2$. If d was not even, then one of $a+d$ and $a+2d$ would be even, but then they could not be prime since they are greater than 2. Thus d must be a multiple of 2 and we must have $a > 2$. If d was not a multiple of 3, then one of $a+d$, $a+2d$ and $a+3d$ would be a multiple of 3, but then they could not be prime since they are greater than 3. Thus d must be a multiple of 3 and we must have $a > 3$. Similarly, d must be a multiple of 5 and we must have $a > 5$. Also similarly, if d is not a multiple of 7, then one of $a, a+d, \cdots, a+6d$ must be a multiple of 7, and since they are all prime with $a \ge 7$ it must be that $a = 7$. Thus either $a = 7$ and d is a multiple of $2\cdot 3\cdot 5 = 30$, or $a > 7$ and d is a multiple of $2\cdot 3\cdot 5\cdot 7 = 210$. If $a > 7$ and d is a multiple of 210 the $a+6d > 7+6\cdot 210 > 900$. If $a = 7$ and $d = 30k$ with $k \in \mathbf{Z}^+$ then we must have $k \le 4$ because when $d = 30\cdot 5 = 150$ we have $a+6d = 7+6\cdot 150 = 907 > 900$. Finally note that when $d = 30\cdot 1 = 30$ we have $a+6d = 187 = 11\cdot 17$, when $d = 30\cdot 2 = 60$ we have $a+3d = 187$, when $d = 30\cdot 3 = 90$ we have $a+2d = 187$, and when $d = 30\cdot 4 = 120$ we have $a+2d = 247 = 13\cdot 19$.

12: Every open set in the plane is equal to the union of a set of disjoint non-degenerate closed line segments.

Solution: This is TRUE. To see this, first convince yourself (or prove) that every open set in $\mathbf{R}^2$ is a union of disjoint half-open rectangles of the form $[a_1,a_2)\times[b_1,b_2) = \big\{(x,y)\in\mathbf{R}^2 \big| a_1 \le x < a_2\,,\, b_1 \le y < b_2\big\}$. For a point $p \in \mathbf{R}^2$ and two vectors $u, v \in \mathbf{R}^2$ with $\{u,v\}$ linearly independent, the closed line segment from p to $p+u$ is the set $[p,p+u] = \big\{p+tu \big| 0 \le t \le 1\big\}$, and we define the *half-open triangle* $[p,p+u,p+v)$ to be the set $[p,p+u,p+v) = \big\{p+su+tv \big| s,t \ge 0\,,\, s+t < 1\big\}$. Note that every half-open rectangle is equal to the union of disjoint half-open triangles, indeed given a half-open rectangle rectangle $[a_1,a_2)\times[b_1,b_2)$, let p,q,r,s be the vertices $p = (a_1,b_1)$, $q = (a_2,b_1)$, $r = (a_2,b_2)$ and $s = (a_1,b_2)$, then choose e sequence of points $p_0, p_1, p_2, \cdots$ increasing vertically with $p_0 = p$ and $\lim_{n\to\infty} p_n = s$, then note that the rectangle is equal to the union $[p_0,q,r)\cup[p_1,p_0,r)\cup[p_2,p_1,r)\cup[p_3,p_2,r)\cup\cdots$. Finally, note that every half-open triangle is equal to the union of disjoint closed line segments, indeed the half-open triangle $[p,p+u,p+v)$ is equal to the union $\big[p, p+\frac{1}{2}u\big] \cup \bigcup\limits_{0<t<1} \big[p + \frac{1+t}{2}\,u\,,\, p+tv\big]$.

13: A rectangular box with sides of length 1, 2 and 3 hovers above the ground. The sun shines down from directly overhead casting a shadow on the horizontal ground. The maximum possible area of the shadow is equal to 7.

Solution: This is TRUE. The rectangular faces of the box have diagonals of lengths $\sqrt{5}$, $\sqrt{10}$ and $\sqrt{13}$. The shadow of the box on the ground, that is the orthogonal projection of the box onto the ground, is a (possibly degenerate) hexagon whose opposite edges are parallel. The area of such a hexagon is twice the area of the triangle on alternate vertices of the hexagon. Thus the area of the shadow of the box is equal to twice the area of the projection of a triangle whose vertices are three of the vertices of the box and whose edges are three of the diagonals of the faces of the box. The area is maximized when the triangle is horizontal so that its orthogonal projection is a triangle of the same shape with sides of lengths $\sqrt{5}$, $\sqrt{10}$ and $\sqrt{13}$. The area of the triangle is easily calculated by placing the vertices of the triangle at $(0,0)$, $(2,1)$ and $(-1,3)$. The area of the triangle is $\frac{7}{2}$, so the area of the hexagonal shadow of the box is 7.