1. Suppose $n = 5\,41G\,507\,2H6$ where G and H are single digits in base 10.
   TRUE or FALSE?
   There are exactly 3 pairs of digits (G, H) such that n is divisible by 72.

TRUE.

For n to be divisible by 8, $H = 1, 5, 9$.

For n to be divisible by 9, the sum of the digits of n is divisible by 9.

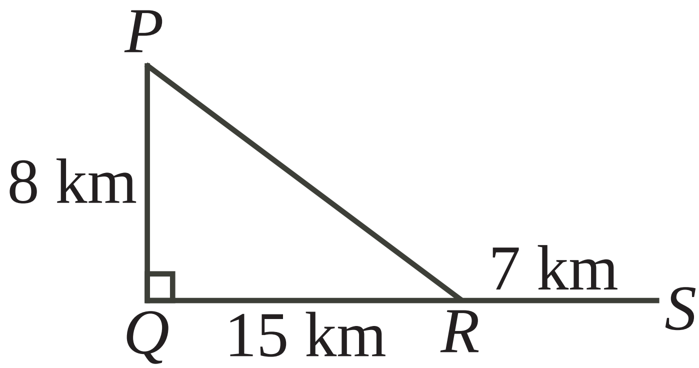
If $H = 1$, then $G = 5$.

If $H = 5$, then $G = 1$.

If $H = 9$, then $G = 6$.

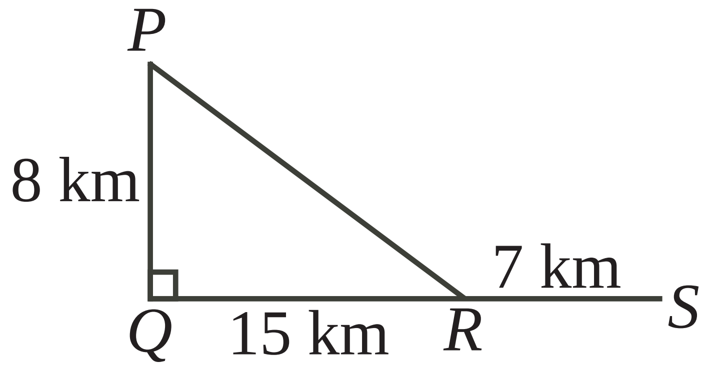
Therefore, there are 3 such pairs.

2. Asafa ran at a speed of 21 km/h from P to Q to R to S, as shown. Florence ran at a constant speed from P directly to R and then to S. They left P at the same time and arrived at S at the same time.



TRUE or FALSE?
Florence arrived at R exactly 7 minutes before Asafa.

FALSE.



By the Pythagorean Theorem,

$$PR = \sqrt{QR^2 + PQ^2} = \sqrt{15^2 + 8^2} = \sqrt{289} = 17 \text{ km}$$

Asafa runs a total distance of $8+15+7 = 30$ km at 21 km/h in the same time that Florence runs a total distance of $17 + 7 = 24$ km.
Therefore, Asafa's speed is $\frac{30}{24} = \frac{5}{4}$ of Florence's speed, so Florence's speed is $\frac{4}{5} \times 21$ or $\frac{84}{5}$ km/h.

Asafa runs the last 7 km in $\frac{7}{21} = \frac{1}{3}$ hour, or 20 minutes.
Florence runs the last 7 km in $\dfrac{7}{\frac{84}{5}} = \dfrac{35}{84} = \dfrac{5}{12}$ hour,

or 25 minutes.
Since Asafa and Florence arrive at S together, then Florence arrived at R 5 minutes before Asafa.

$2\frac{1}{2}$. A "look and see" sequence starts with a given sequence of digits on Day 0.

On Day k, the new sequence is formed by reading the sequence on Day $k-1$ in groups of like numbers.

For example, if the sequence on Day 0 is 111221, the sequence on Day 1 is "three ones two twos one one" or 312211, and the sequence on Day 2 is 13112221, and so on.

TRUE or FALSE?

If neither the Day 0 sequence nor the Day 1 sequence contains the digit 4, then none of the sequences on any Day can contain the digit 4.

(Corrected solution)
FALSE.
Suppose the Day 0 sequence consists of 111 digits, each of which is 1.
Then the Day 1 sequence is 1111 (“one hundred eleven 1’s”) and so the Day 2 sequence is 41.

3. The weight of a matrix is its number of non-zero entries.
TRUE or FALSE?
The number of 15 by 16 matrices with entries from $\mathbb{Z}_{17}$ of rank 2 and weight 2 is larger than 6452100.

FALSE.

For a $m \times n$ matrix to have rank 2 and weight 2, it must have 2 entries which are non-zero and which are in different rows and columns.

There are mn positions in which the first entry can be placed and then $(m-1)(n-1)$ positions where the second entry can be placed (all but those in the row and column of the first entry).

By multiplying the number of possible positions together, we double-count as each pair can be chosen in two orders. Therefore the number of choices of two positions is

$$\tfrac{1}{2}mn(m-1)(n-1)$$

There are 16 choices for each of the non-zero entries. Therefore, the number of such matrices is

$$\begin{aligned}\tfrac{1}{2}(15)(16)(14)(15)(16)(16) &= 120(210)(256) \\ &= 25200(256) \\ &= 6451200\end{aligned}$$

4. Let R_n be the region in the first quadrant bounded by $y = f(x) = x^n$ and $y = g(x) = \sqrt[n]{x}$.
TRUE or FALSE?
As $n \to \infty$, the centroid of R_n approaches $\left(\frac{1}{e}, \frac{1}{e}\right)$.

Note:
The x-coordinate, $\overline{x}$, of the centroid of a region R with area A bounded by $y = f(x)$ and $y = g(x)$ which intersect at $x = a$ and $x = b$ is given by

$$\overline{x} = \frac{1}{A} \int_a^b x(f(x) - g(x))\, dx$$

The y-coordinate is defined analogously.

FALSE.
The boundary curves intersect at $x = 0$ and $x = 1$.
The area of region R_n is

$$\int_0^1 (x^{1/n} - x^n)\,dx = \left(\frac{n}{n+1}x^{(n+1)/n} - \frac{1}{n+1}x^{n+1}\right)\Bigg|_0^1$$

which equals $\dfrac{n-1}{n+1}$.
The x-coordinate of the centroid is thus

$$\begin{aligned}
&\frac{n+1}{n-1}\int_0^1 x(x^{1/n} - x^n)\,dx \\
&\qquad = \frac{n+1}{n-1}\left(\frac{n}{2n+1}x^{(2n+1)/n} - \frac{1}{n+2}x^{n+2}\right)\Bigg|_0^1 \\
&\qquad = \frac{n+1}{n-1}\left(\frac{n}{2n+1} - \frac{1}{n+2}\right) \\
&\qquad = \frac{(n+1)(n^2-1)}{(n-1)(2n+1)(n+2)} \\
&\qquad = \frac{(n+1)(n+1)}{(2n+1)(n+2)}
\end{aligned}$$

As $n \to \infty$, this coordinate approaches $\frac{1}{2}$, not $\frac{1}{e}$.

(Alternatively, we could note that, as $n \to \infty$, the region approaches the unit square, so the centroid should approach $(\frac{1}{2}, \frac{1}{2})$.)

5. TRUE or FALSE?
   There exist 803 disjoint unordered pairs of distinct positive integers with distinct sums at most 2007.

FALSE.

Suppose that there are N disjoint pairs of distinct positive integers with distinct sums at most 2007.

Then the sum of integers in the N pairs is at least

$$1 + 2 + \cdots + 2N = \tfrac{1}{2}(2N)(2N+1)$$

(as the smallest the integers could be is the set of the integers 1 to $2N$), and is at most

$$2007 + 2006 + \cdots + (2007 - N + 1) = \tfrac{1}{2}N(4015 - N)$$

(as the most the sums of the pairs could be are 2007, 2006 and so on down to $2007 - N + 1$).

Therefore,

$$\begin{aligned} \tfrac{1}{2}(2N)(2N+1) &\leq \tfrac{1}{2}N(4015 - N) \\ 4N + 2 &\leq 4015 - N \\ 5N &\leq 4013 \end{aligned}$$

so $N \leq 802$.

(This is sharp as the pairs

$$(2, 1203), (4, 1202), \ldots, (802, 803)$$

$$(1, 1605), (3, 1604), \ldots, (801, 1205)$$

demonstrate.)

6. Let W be the set of finite products of the rational numbers of the form $\dfrac{3n+2}{2n+1}$ with $n \geq 0$.
TRUE or FALSE?
$5 \notin W$

TRUE.

First, we note that for $n \geq 0$, we have $\dfrac{3n+2}{2n+1} \geq \dfrac{3}{2}$.

Also, $\left(\frac{3}{2}\right)^4 = \frac{81}{16} > 5$.

Thus, if $5 \in W$, it is made from at most 3 of the "basic" fractions, as the product of 4 or more basic fractions is too large.

If each of the (at most) three fractions is $\frac{5}{3}$ or smaller (that is, has $n \geq 1$), then their product is at most $\frac{125}{27} < 5$.

Therefore, if $5 \in W$, then its representation must include at least one copy of the $n = 0$ fraction (that is, 2).

But in that case, the numerator of the product is even, and the denominator is odd, so the product cannot equal 5.

7. Define $f : [0,1] \times [0,1] \longrightarrow [0,1] \times [0,1]$ by

$$f(x,y) = \left(\tfrac{1}{2}(x+y), \sqrt{xy}\right)$$

TRUE or FALSE?
The area of the image of f is $\frac{1}{6}$.

TRUE.

Suppose $(p, q) = f(x, y)$.

By the AM-GM inequality, $p \geq q$.

For what $(p, q) \in [0, 1]^2$ can we actually solve for $(x, y) \in [0, 1]^2$?

If $\frac{1}{2}(x+y) = p$ and $\sqrt{xy} = q$, then $x+y = 2p$ and $xy = q^2$, so

$$\begin{aligned} x(2p - x) &= q^2 \\ 0 &= x^2 - 2px + q^2 \\ x &= p \pm \sqrt{p^2 - q^2} \end{aligned}$$

by the quadratic formula.

(x and y will be of this form, taking the opposite signs. We want $x, y \in [0, 1]$.)

Clearly $0 = p - p \leq p - \sqrt{p^2 - q^2} \leq p \leq 1$.

Also, $0 \leq p + \sqrt{p^2 - q^2}$.

For $p + \sqrt{p^2 - q^2} \leq 1$, we have $\sqrt{p^2 - q^2} \leq 1 - p$ or $p \leq \frac{1}{2}(1 + q^2)$.

Therefore, in the pq-plane, the solution set lies below the line $q = p$, above the p-axis, and to the left of the parabola $p = \frac{1}{2}(1 + q^2)$.

Therefore, the area of the image is

$$\int_0^1 \left(\tfrac{1}{2}(1 + q^2) - q\right) \, dq = \tfrac{1}{2}q + \tfrac{1}{6}q^2 - \tfrac{1}{2}q^2\Big|_0^1 = \tfrac{1}{6}$$

8. TRUE or FALSE?
   There exist two non-congruent triangles of equal perimeter and equal area.

TRUE.

The right-angled triangle with sides of lengths 3, 4 and 5 has perimeter 12 and area 6.

Consider an isosceles triangle with sides of length x, x and $12 - 2x$ (and thus perimeter 12).

Using Heron's formula to calculate the area,

$$\begin{aligned}
\sqrt{6(6-x)(6-x)(6-(12-2x))} &= 6 \\
6(2x-6)(x-6)^2 &= 36 \\
(x-3)(6-x)^2 &= 3
\end{aligned}$$

When $x = 3$ (giving a 3-3-6 "triangle"), the left side equals 0.

When $x = 4$ (giving a 4-4-4 triangle), the left equals 4.

Therefore, between $x = 3$ and $x = 4$, there is a solution, and thus a triangle of the required area and perimeter (and it is indeed a triangle!).

Alternatively, we could consider an isosceles triangle with height h, base $\dfrac{12}{h}$ and thus area 6.

Its perimeter is $p(h) = \dfrac{12}{h} + 2\sqrt{h^2 + \dfrac{36}{h^2}}$.

Since $p(3) = 4 + 2\sqrt{13} < 12$ and $p(h) \to \infty$ as $h \to \infty$, there must be an h_0 for which $p(h_0) = 12$ by the Intermediate Value Theorem.

9. Let $\sum_{n=1}^{\infty} \frac{a_n}{n}$ be an infinite sum with cycling coefficients
a_1, a_2, a_3, a_4.
(That is, $a_n = a_1$ if $n \equiv 1 \pmod 4$, and so on.)
TRUE or FALSE?
The sum converges if and only if $a_1 + a_2 + a_3 + a_4 = 0$.

TRUE.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a_n}{n} &= \sum_{k=1}^{\infty}\left(\frac{a_1}{4k-3}+\frac{a_2}{4k-2}+\frac{a_3}{4k-1}+\frac{a_4}{4k}\right)\\ &= \sum_{k=1}^{\infty}\frac{64(a_1+a_2+a_3+a_4)k^3+(\cdot)k^2+(\cdot)k+(\cdot)}{4k(4k-1)(4k-2)(4k-3)}\end{aligned}$$

By the Comparison Test, this sum converges if and only if the coefficient of k^3 in the numerator equals 0, that is, if and only if $a_1+a_2+a_3+a_4=0$.

10. Suppose that the 52 cards in a deck are numbered from 1 to 52 (from top to bottom). Lino then perfectly shuffles the deck is 2007 times. After removing the bandages from his blistered fingers, Lino looks for the card numbered 42 in the deck.

    TRUE or FALSE?

    Card 42 is in position 34 (counting from the top).

    Note:

    Each time Lino perfectly shuffles the deck, he takes the top 26 cards and interleaves them with the bottom 26 cards, starting with a card from the bottom half.

    Thus, after the first shuffle, the order of cards is 27, 1, 28, 2, and so on.

TRUE.

After the first shuffle, card x is in position $2x \pmod{53}$.

Therefore, after k shuffles, card x will be in position $2^k x \pmod{53}$.

Thus, we must calculate $2^{2007}(42) \pmod{53}$.

Since 53 is prime, $2^{52} \equiv 1 \pmod{53}$ by Fermat's Theorem.

Thus, $2^{1876} \equiv 1 \pmod{53}$, since $1876 = 38(52)$.

So we must calculate $2^{31}(42) \pmod{53}$.

Now $2^{10} = 1024 \equiv 17 \pmod{53}$, so

$$\begin{aligned} 2^{31}(42) &\equiv 2(17)(17)(17)(42) \pmod{53} \\ &\equiv 289(84)(17) \pmod{53} \\ &\equiv 24(31)(17) \pmod{53} \\ &\equiv 744(17) \pmod{53} \\ &\equiv 2(17) \pmod{53} \\ &\equiv 34 \pmod{53} \end{aligned}$$

Therefore, card 42 is in position 34.

11. Ken quickly calculates that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.

Ken never makes mistakes.

TRUE or FALSE?

$$\sum_{\substack{n \text{ cubefree} \\ n \geq 1}} \frac{1}{n^2} \leq 1.575.$$

Note:

A positive integer is called "cubefree" if it is not exactly divisible by the cube of any positive integer larger than 1. For example, 12 is cubefree but 24 is not.

FALSE.

Let $S = \displaystyle\sum_{\substack{n \text{ cubefree} \\ n \geq 1}} \frac{1}{n^2}$.

Then

$$
\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{\substack{n \text{ not cubefree} \\ n \geq 1}} \frac{1}{n^2} \\
S &= \frac{\pi^2}{6} - \left[\left(\frac{1}{8^2} + \frac{1}{27^2} + \frac{1}{64^2} + \cdots \right) \sum_{\substack{n \text{ cubefree} \\ n \geq 1}} \frac{1}{n^2} \right] \\
&\qquad \text{(since every positive integer that is not cubefree} \\
&\qquad \text{can be uniquely written as the product of a cube} \\
&\qquad \text{and a cubefree integer)} \\
S &= \frac{\pi^2}{6} - \left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots \right) S \\
S &= \frac{\pi^2}{6} - \left(\frac{\pi^6}{945} - 1 \right) S \\
\frac{\pi^6}{945} S &= \frac{\pi^2}{6} \\
S &= \frac{945}{6\pi^4} = \frac{315}{2\pi^4}
\end{aligned}
$$

Since $\pi^2 < 10$, then $\pi^4 < 100$, so $S > \frac{315}{200} = 1.575$.

12. TRUE or FALSE?
There exists a continuous function $f : [0, 1] \to [0, 1]$ with the property that for every $y \in [0, 1]$, there exist infinitely many values of x such that $f(x) = y$.

TRUE.
Let $h : [0,1] \longrightarrow [0,1] \times [0,1]$ given by $h(t) = (f(t), g(t))$ be a space-filling curve.
Then $f(t)$ has the desired property.