1. Let $f(x) = \ln\left(\sqrt{1+e^{\cos x}}\right)$.
   TRUE or FALSE?
   $f(0)f'(0)f''(0) = -1$

FALSE.
First, $f(0) = \ln\left(\sqrt{1+e^1}\right) = \ln\left(\sqrt{1+e}\right)$.
Next,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\ln\left(\sqrt{1+e^{\cos x}}\right)\right) \\ &= \frac{d}{dx}\left(\tfrac{1}{2}\ln\left(1+e^{\cos x}\right)\right) \\ &= \tfrac{1}{2}\cdot\frac{1}{1+e^{\cos x}}\cdot e^{\cos x}\cdot(-\sin x) \end{aligned}$$

Substituting $x = 0$, $f'(0) = \dfrac{e\cdot 0}{2(1+e)} = 0$.
Therefore, $f(0)f'(0)f''(0) = 0$.

2. Let $S = 1 + 2 + 3 + \cdots + 2005 + 2006$.
TRUE or FALSE?
S has exactly 4 distinct prime factors.

(N.B. If 2 was a prime factor of S twice (which it isn't), it would be counted as only one distinct prime factor.)

TRUE.

$$\begin{aligned} 1 + 2 + \cdots + 2005 + 2006 &= \tfrac{1}{2}(2006)(2007) \\ &= (1003)(2007) \\ &= (17)(59)(3^2)(223) \end{aligned}$$

Each of 3, 17, 59, and 223 is prime.

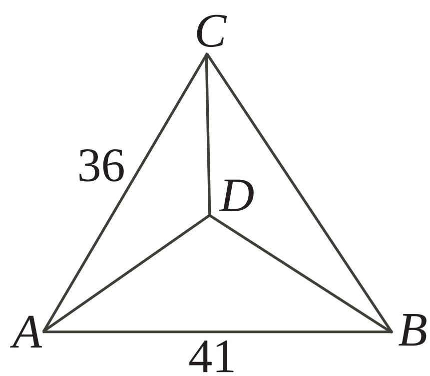
3. The lengths of the six edges of tetrahedron $ABCD$ are 7, 13, 18, 27, 36, and 41.
TRUE or FALSE?
If $AB = 41$, it is possible to construct tetrahedron with $CD = 7$.

FALSE.

Consider triangles CAB and DAB.

The length of one of the sides of one of these triangles must be 36, since we cannot divide 7, 13, 18, 27 into pairs, the sum of each of which is more than 41.

Suppose $CA = 36$.



The two remaining edges in $\triangle DAB$ must have lengths 27 and 18, since their sum is more than 41.

Consider $\triangle DAC$. The sum of the lengths of edges AD and CD must be more than 36, so we must have $AD = 27$ (and so $BD = 18$) and $CD = 13$.

4. TRUE or FALSE?
The equation

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$$

has exactly 8 real solutions.

FALSE.
Notice that $(\sqrt{x-1}-2)^2 = x+3-4\sqrt{x-1}$ and $(\sqrt{x-1}-3)^2 = x+8-6\sqrt{x-1}$.
Therefore, the equation becomes

$$|\sqrt{x-1}-2| + |\sqrt{x-1}-3| = 1$$

which is satisfied by every real number in $[5, 10]$.

5. Suppose $(1 + x + x^2)^{50} = a_0 + a_1 x + \cdots + a_{100} x^{100}$.
TRUE or FALSE?
$a_0 + a_2 + \cdots + a_{98} + a_{100}$ is odd.

TRUE.

Substituting $x = 1$, we obtain

$$3^{50} = a_0 + a_1 + a_2 + \cdots + a_{99} + a_{100}$$

Substituting $x = -1$, we obtain

$$1^{50} = a_0 - a_1 + a_2 - \cdots + a_{98} - a_{99} + a_{100}$$

Therefore,

$$3^{50} + 1 = 2(a_0 + a_2 + \cdots + a_{98} + a_{100})$$

or

$$a_0 + a_2 + \cdots + a_{98} + a_{100} = \frac{3^{50} + 1}{2}$$

Since $3^{50} \equiv 1 \pmod 4$, then the right side is odd, and so $a_0 + a_2 + \cdots + a_{98} + a_{100}$ is odd.

6. Let A be the statement "$14! = 87278291200$".

Let B be the statement "The minimum value for the sum of the x- and y-intercepts of a line with negative slope passing through $(3, 12)$ is 27."

Let C be the statement "There is exactly one point on the Earth with the property that when Tom starts at that point, travels 100 km south, then 100 km east, then 100 km north, he ends up at the point where he started."

TRUE or FALSE?
A is FALSE and B is TRUE and C is TRUE.

FALSE.
The sum of the digits of 87278291200 is 46, so 87278291200 is not divisible by 3.

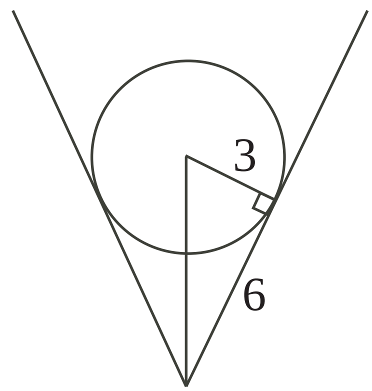
Suppose that the line has slope $-m$ with $m > 0$, and so has equation $y = -mx + (3m + 12)$.
Therefore, the y-intercept of the line is $3m + 12$ and its x-intercept is $\dfrac{3m+12}{m}$.
Define $f(m) = 3m + 12 + \dfrac{3m+12}{m} = 3m + 15 + \dfrac{12}{m}$.
The sum $3m + \dfrac{12}{m}$ is minimized when $m = 2$ (using either calculus or the AM-GM inequality).
Therefore, the minimum value is $f(2) = 27$.

The north pole has this property, as does any point 100 km north of the latitude whose latitude circle has length 100 km.

7. A heavy ball of radius 3 is placed into a cup, which is in the shape of an inverted cone of radius 5 and height 10. TRUE or FALSE?
   When the cup is filled to the brim with water, the ball is totally submerged.

TRUE.

The diagram shows a cross-section through the centre of the sphere and the axis of the cone. Because the ratio of radius to height of the cone is $1 : 2$, then the tangent of half of the vertex angle is $\frac{1}{2}$.



The distance from the centre of the sphere to the tip of the cone is $\sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$, so the distance from the tip of the cone to the top of the sphere is $3\sqrt{5} + 3$.

Since $\sqrt{5} < \frac{7}{3}$ (as $5 < \frac{49}{9}$) then the distance from the tip of the cone to the top of the sphere is less than 10, so the sphere is submerged.

$7\frac{1}{2}$. TRUE or FALSE?

L'Hôpital's Rule was discovered by James Bernoulli.

FALSE.
It was discovered by John Bernoulli.

8. Define $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.
TRUE or FALSE?

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{f_k f_{k+1}} < \frac{1}{\sqrt{5}}$$

TRUE.
Using the well-known (!) property that

$$f_{k-1}f_{k+1} - f_k^2 = (-1)^k$$

for $k \geq 2$,

$$\begin{aligned}
1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{f_k f_{k+1}} &= 1 + \lim_{N\to\infty} \sum_{k=1}^{N} \frac{(-1)^k}{f_k f_{k+1}} \\
&= 1 - \frac{1}{f_1 f_2} + \lim_{N\to\infty} \sum_{k=2}^{N} \frac{f_{k-1}f_{k+1} - f_k^2}{f_k f_{k+1}} \\
&= 1 - \frac{1}{f_1 f_2} + \lim_{N\to\infty} \sum_{k=2}^{N} \left[\frac{f_{k-1}}{f_k} - \frac{f_k}{f_{k+1}}\right] \\
&= 1 - \frac{1}{f_1 f_2} + \frac{f_1}{f_2} - \lim_{N\to\infty} \frac{f_N}{f_{N+1}} \\
&= 1 - 1 + 1 - \frac{2}{1+\sqrt{5}} \\
&= 1 - \frac{\sqrt{5}-1}{2} \\
&= \frac{3-\sqrt{5}}{2}
\end{aligned}$$

and

$$\frac{3-\sqrt{5}}{2} < \frac{1}{\sqrt{5}} \Leftrightarrow 3\sqrt{5} - 5 < 2 \Leftrightarrow 3\sqrt{5} < 7 \Leftrightarrow 45 < 49$$

9. A positive integer N in base 10 has fewer than 25 digits and begins with the digits 15 on the left. When multiplied by 5, the only change in the number is to shift the digits 15 to the right hand end of the integer.
   TRUE or FALSE?
   N has 16 digits

FALSE.

Suppose that N has $k+2$ digits, so is equal to $15\times 10^k+M$ for some k digit positive integer M.

Then $5N = 75\times 10^k+5M$ and this must equal $100M+15$, so $15 \times 10^k + M = 20M + 3$ or $15 \times 10^k = 19M + 3$.

Thus $15 \times 10^k \equiv 3 \pmod{19}$ or $10^k \equiv 4 \pmod{19}$.

Since $k < 25$, k must equal 16, so N has 18 digits.

For the record, $N = 157\,894\,736\,842\,105\,263$.

10. For p a real number, define $s_k(p) = \sum_{n=1}^{k} \frac{1}{n^p}$.

TRUE or FALSE?
If $p < 1$,

$$\lim_{n\to\infty} \frac{s_{2n}(p)}{s_n(p)} = 2^{-p}$$

FALSE.

For $p < 1$, then looking at upper and lower sums

$$\int_0^m \frac{1}{x^p} dx > s_m(p) > \int_1^{m+1} \frac{1}{x^p} dx$$

and so

$$\frac{1}{1-p} m^{1-p} > s_m(p) > \frac{1}{1-p}((m+1)^{1-p} - 1)$$

and so

$$\frac{(2n)^{1-p}}{(n+1)^{1-p} - 1} > \frac{s_{2n}(p)}{s_n(p)} > \frac{(2n+1)^{1-p} - 1}{n^{1-p}}$$

By the Squeeze Theorem,

$$\lim_{n \to \infty} \frac{s_{2n}(p)}{s_n(p)} = 2^{1-p}$$

11. Larry and Barry are playing tennis. When Larry is serving, the probability that he wins a given point is $\frac{2}{3}$ because of his sinister techniques.
TRUE or FALSE?
If P is the probability that Larry wins a game where he is serving, then $\left|P - \frac{6}{7}\right| \geq \frac{1}{850}$.

(NOTE: In a game in tennis, the same player serves each point. To win the game, a player must be the first player to win four points and must win by two points. If the game is tied 3-3, the game continues until one player wins by two points.)

FALSE.
Suppose that the probability that Larry wins a point is p.
The probability that Larry wins 4-0 is p^4.
The probability that Larry wins 4-1 is $\binom{4}{1} p^4(1-p)$ since Barry can win any one of the first four points of the game (but not the last point).
The probability that Larry wins 4-2 is $\binom{5}{2} p^4(1-p)^2$.
Otherwise, the game reaches a 3-3 tie (with probability $\binom{6}{3} p^3(1-p)^3$) and each pair of points is alternated until Larry wins two points in a row.
In this case, the sequence of point winners beyond 3-3 could be LL, BLLL, LBLL, etc., so the probability beyond the 3-3 stage is

$$p^2 + 2^1 p(1-p)p^2 + 2^2 p^2(1-p)^2 p^2 + \cdots$$

since there are two possible orders for each pair.
Thus, the probability that the game is won beyond the 3-3 stage is

$$\binom{6}{3} p^3(1-p)^3 \left(\frac{p^2}{1-2p(1-p)}\right)$$

Therefore, the probability is

$$p^4 + 4p^4(1-p) + 10p^4(1-p)^2 + \frac{20p^5(1-p)^3}{2p^2-2p+1}$$

When $p = \frac{2}{3}$, we obtain $\frac{16}{81} + \frac{64}{243} + \frac{160}{729} + \frac{640/6561}{5/9} = \frac{624}{729} = \frac{208}{243}$.
So $\left|\frac{208}{243} - \frac{6}{7}\right| = \frac{2}{1701} < \frac{2}{1700} = \frac{1}{850}$.

12. TRUE or FALSE?
The Diophantine Equation $a^3+b^3 = c^4$ has infinitely many solutions with c odd.

TRUE.

Consider $n^3 + (n+1)^3 = 2n^3 + 3n^2 + 3n + 1$, which must be odd, since n and $n+1$ have opposite parity.

Set $N = 2n^3 + 3n^2 + 3n + 1$.

Then $n^3 + (n+1)^3 = N$ implies $(nN)^3 + (N(n+1))^3 = N^4$.

Since N is odd and there are infinitely many such solutions, then the statement is TRUE.

13. Let $p(x)$ be a monic quadratic polynomial with integral roots -1 and r.
TRUE or FALSE?
There are exactly two values of r for which $p(p(x)) = 0$ has exactly 3 distinct real roots.

FALSE.
Since the roots of $p(x)$ are -1 and r, then
$p(x) = x^2 + (1-r)x - r$.
If $p(p(x)) = 0$ then $p(x) = r$ or $p(x) = -1$.
If $p(x) = r$, then $x^2 + (1-r)x - 2r = 0$, which has discriminant $\Delta_1 = (1-r)^2 + 8r = r^2 + 6r + 1$.
If $p(x) = -1$, then $x^2 + (1-r)x + (1-r) = 0$, which has discriminant $\Delta_2 = (1-r)^2 - 4(1-r) = r^2 + 2r - 3$.
There are no integer values of r for which $r^2 + 6r + 1 = 0$, so the first equation cannot have a repeated root.
There are two integer values of r (namely 1 and -3) for which $r^2+2r-3 = 0$, ie. for which the second equation has a repeated root. However, if $r = -3$, the first equation has no real roots. If $r = 1$, the first equation has two distinct real roots and the second has one real root.
Can the two equations have a common root? If they had a common root a, then

$$a^2 + (1-r)a - 2r = a^2 + (1-r)a + (1-r)$$

or $r = -1$. (Notice that when $r = -1$, though, neither equation has a real root.)
So when $r = 1$, there are indeed three distinct real roots.
Thus, there is one value of r for which $p(p(x)) = 0$ has 3 distinct real roots.

14. TRUE or FALSE?

$$\lim_{n\to\infty} e^{\sin(1/1)+\sin(1/2)+\cdots+\sin(1/n)+1} - e^{\sin(1/1)+\sin(1/2)+\cdots+\sin(1/n)}$$

exists.

FALSE.

$$
\begin{aligned}
& e^{\sin(1/1)+\sin(1/2)+\cdots+\sin(1/n)+1} - e^{\sin(1/1)+\sin(1/2)+\cdots+\sin(1/n)} \\
= \ & e^{\sin(1/1)+\sin(1/2)+\cdots+\sin(1/n)}(e-1)
\end{aligned}
$$

Since the series $\displaystyle\sum_{n=1}^{\infty} \sin(1/n)$ diverges (use the Limit Comparision Test with the series $\displaystyle\sum_{n=1}^{\infty} 1/n$ which diverges), then the limit does not exist.

$14\frac{1}{2}$. For p a prime number, define $f : \mathbb{Z}_{p^2} \longrightarrow \mathbb{Z}_{p^2}$ by $f(0) = 0$ and $f(a) = 1$ for $a \neq 0$.

TRUE or FALSE?

There does not exist a p for which f can be written as a polynomial with coefficients in $\mathbb{Z}_{p^2}$.

TRUE.
Let n be a composite positive integer. Then $n = qr$ for some positive integers $1 < q, r < n$.
Suppose that f could be written as a polynomial

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$$

with $a_m, a_{m-1}, \ldots, a_1, a_0 \in \mathbb{Z}_n$.
Since $f(0) = 0$, then $a_0 = 0$ in $\mathbb{Z}_n$.
Then $f(q) = a_m q^m + a_{m-1} q^{m-1} + \cdots + a_1 q$ is divisible by q so cannot be of the form $bn + 1$ (that is, cannot be congruent to 1 modulo n) since $q \mid n$.
This is a contradiction.

15. Define $P_N(x) = \displaystyle\sum_{k=0}^{N} \frac{x^k}{k!}$.

TRUE or FALSE?

There does not exist a positive integer N for which $P_N(x)$ has a repeated root.

TRUE.

For a polynomial $f(x)$ to have a repeated root, $\gcd(f(x), f'(x))$ must not be constant.

Here, $P_N'(x) = \dfrac{d}{dx}\left(\displaystyle\sum_{k=0}^{N} \frac{x^k}{k!}\right) = \displaystyle\sum_{k=1}^{N} \frac{kx^{k-1}}{k!} = P_{N-1}(x)$.

Thus,

$$\begin{aligned} \gcd(P_N(x), P_N'(x)) &= \gcd(P_N(x), P_{N-1}(x)) \\ &= \gcd(P_N(x), P_N(x) - P_{N-1}(x)) \\ &= \gcd(P_N(x), \tfrac{1}{N!}x^N) \end{aligned}$$

Since 0 is not a root of $P_N(x)$, then the gcd is 1, so there is no repeated root.

16. TRUE or FALSE?

$$\lim_{x \to 0} \frac{\sin(\tan x) - \tan(\sin x)}{x^7} > \frac{\pi}{24}$$

TRUE.

Let $f(x) = x + a_3x^3 + a_5x^5 + a_7x^7 + \ldots$ and $g(x) = x + b_3x^3 + b_5x^5 + b_7x^7 + \ldots$ be two power series which converge in a neighbourhood of 0.

Then

$$\begin{aligned} f(g(x)) \;=\; & x + (a_3 + b_3)x^3 + \\ & (a_5 + b_5 + 3a_3b_3)x^5 \\ & (a_7 + b_7 + 3a_3b_5 + 5a_5b_3 + 3a_3b_3^2)x^7 + \ldots \end{aligned}$$

Thus

$$\frac{f(g(x)) - g(f(x))}{x^7} = 2(a_5b_3 - a_3b_5) + 3a_3b_3(b_3 - a_3) + O(x)$$

With $f(x) = \sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \ldots$ and $g(x) = \tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$, we obtain the desired limit to be

$$2\left(\left(\frac{1}{120}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{6}\right)\left(\frac{2}{15}\right)\right) + 3\left(\frac{-1}{6}\right)\left(\frac{1}{3}\right)\left(\frac{-1}{2}\right) = \frac{2}{15} > \frac{\pi}{24}$$