Principally unimodular skew-symmetric matrices

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Abstract

A square matrix is *principally unimodular* if every principal submatrix has determinant 0 or ± 1. Let A be a symmetric $(0,1)$-matrix, with a zero diagonal. A *PU-orientation* of A is a skew-symmetric signing of A that is PU. If A' is a PU-orientation of A, then, by a certain decomposition of A, we can construct every PU-orientation of A from A'. This construction is based on the fact that the PU-orientations of indecomposable matrices are unique up to negation and multiplication of certain rows and corresponding columns by -1. This generalizes the well-known result of Camion, that if a $(0,1)$-matrix can be signed to be totally unimodular then the signing is unique up to multiplying certain rows and columns by -1. Camion's result is an easy but crucial step in proving Tutte's famous excluded minor characterization of totally unimodular matrices.

1 Introduction

A square matrix A is called *principally unimodular (PU)* if every nonsingular principal submatrix is unimodular (that is, has determinant ± 1). Let A be a symmetric $(0,1)$-matrix, with a zero diagonal, a skew-symmetric signing of A is called an *orientation* of A. We are concerned with the orientations of A that are PU; such orientations are called *PU-orientations*, and were initially introduced in relation to circle graphs [3, 6].

Let A be a symmetric $(0,1)$-matrix whose rows and columns are indexed by the set V, and let A' be a PU-orientation of A. We can construct other PU-orientations of A from A', for instance, $-A'$ is PU, we call this construction *negation*. Also, for $X \subseteq V$, the matrix $\left(\begin{array}{c|c} A'[X] & -A'[X, V\setminus X] \\ \hline -A'[V\setminus X, X] & A'[V \setminus X] \end{array}\right)$ is PU (where $A[X,Y]$ denotes the submatrix of A indexed by the rows X and columns Y, and $A[X]$ denotes the principal submatrix $A[X,X]$); this operation is called *cut-switching*. Collectively, we refer to negation and cut-switching as *switching*.

It is not, in general, the case that every two PU-orientations of A are equivalent under switching; for instance, we show that the matrix $J_n - I_n$ has $(n-1)!/2$ distinct PU-orientations, where J_n denotes the $n \times n$ all ones matrix, and I_n denotes the $n \times n$ identity. Let X, Y be a partition of V with $|X|, |Y| \geq 2$, we call (X, Y) a *split* of A if the rank of $A[X, Y]$ is at most 1, a matrix without a split is called *prime*. Our main result is:

Theorem 1.1 *Let A be a symmetric $(0,1)$–matrix with a zero diagonal. If A is prime then every two PU-orientations of A are equivalent under switching.*

As a corollary of Theorem 1.1 we derive a formula for the number of PU-orientations of A distinct up to switching, assuming that A has a PU-orientation; this formula is based on a decomposition of A using certain splits.

Theorem 1.1 is a generalization of a theorem about totally unimodular matrices: a matrix B is totally unimodular if and only if the matrix $\left(\begin{array}{c|c} 0 & B \\ \hline -B^T & 0 \end{array}\right)$ is PU.

Theorem 1.2 (Camion [8]) *If A is a matrix which can be signed to be totally unimodular then such a signing is unique up to multiplication of certain rows and columns by -1.* □

Theorem 1.2 is easy to prove, though it is an important step in proofs of Tutte's famous excluded minor characterization of totally unimodular matrices [20, 21, 14].

Our proof of Theorem 1.1 gives rise to a polynomial-time algorithm for the following problem: *Given a symmetric $(0,1)$-matrix A with a zero diagonal that admits a PU-orientation, find a PU-orientation of A.* Such an algorithm implies that the following problems are algorithmically equivalent, in the sense that (Q_1) is polynomial-time solvable if and only if (Q_2) is polynomial-time solvable. (This equivalence is used in algorithms that recognize total unimodularity.)

(Q_1) Given a symmetric $(0,1)$-matrix A with a zero diagonal, does A admit a PU-orientation?

(Q_2) Given a skew-symmetric matrix A, is A PU?

Delta-matroids

While delta-matroids do not play a role in the proof of Theorem 1.1, the theorem is naturally described in this setting, so we begin by introducing delta-matroids.

Let A be a square matrix with entries defined over a field F, and whose rows and columns are both indexed by V. Define $\mathcal{F}_A = \{S \subseteq V : A[S] \text{ is nonsingular}\}$; by convention we assume $\emptyset \in \mathcal{F}_A$. If A is either symmetric or skew-symmetric then $\mathcal{F}_A$ satisfies the Symmetric Exchange Axiom [4]:

(SEA) For $X, Y \in \mathcal{F}$ and $x \in X \Delta Y$ there exists $y \in X \Delta Y$ such that $X \Delta \{x, y\} \in \mathcal{F}$, where $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$. If $\mathcal{F}$ is a nonempty collection of subsets of V and $\mathcal{F}$ satisfies the (SEA) then $M = (V, \mathcal{F})$ is a *delta-matroid* (see [1]); delta-matroids arising from symmetric and skew-symmetric matrices are called *representable* (see [4]). A delta-matroid that can be represented by a skew-symmetric PU-matrix is called *regular.*

If A is a skew-symmetric matrix, then all sets in $\mathcal{F}_A$ have even cardinality. A delta-matroid $(V, \mathcal{F})$ is called *even* if $|F_1 \Delta F_2|$ is even for all $F_1, F_2 \in \mathcal{F}$.

Let $M = (V, \mathcal{B})$ be a matroid representable over a field F. Here $\mathcal{B}$ is the set of bases of M. For a basis B of M define a matrix A, whose columns are indexed by the set $V \setminus B$, such that $[I|A]$ is a linear representation of M over F. If $[I|A]$ is a representation of M over the reals and A is totally unimodular then M is *regular.* Define $A' = \left(\begin{array}{c|c} 0 & A \\ \hline \pm A^T & 0 \end{array}\right)$. It can be easily verified that A' is PU if and only if A is totally unimodular, and $\mathcal{F}_{A'} = \mathcal{B} \Delta B$, where $\mathcal{B} \Delta B = \{B' \Delta B : B' \in \mathcal{B}\}$. It is also easy to show that, for a collection of subsets $\mathcal{F}$ of V and a set $S \subseteq V$, $\mathcal{F}$ satisfies (SEA) if and only if $\mathcal{F} \Delta S$ satisfies (SEA) (see [7]); this operation is called *twisting.* Two delta-matroids equivalent under twisting are considered to be equivalent, so representability and regularity in delta-matroids naturally generalize their counterparts in matroids.

As is the case with matroids, regularity seems fundamental in the study of representability.

Theorem 1.3 (Geelen [12]) *For an even delta-matroid M, the following are equivalent:*

(i) M is regular,

(ii) M is representable over every field,

(iii) M is representable over both $GF(2)$ and $GF(3)$. □

Our proof of Theorem 1.1 can be generalized to prove the following: *Given a 3-connected delta-matroid M, any two skew-symmetric $GF(3)$-representations of M are switching-equivalent.* For the definition of "3-connectivity", we refer the reader to Bouchet [5]. The requirements for unique $GF(3)$-representability in even delta-matroids are remarkably similar to the requirements for unique $GF(4)$-representability in matroids, see Kahn [17]. In this paper we introduce a tool, called a blocking sequence, for studying splits and prime graphs. Blocking sequences have recently been seen to apply to matroid connectivity, and play a vital role in proving the excluded minor characterization of $GF(4)$-representable matroids of Geelen, Gerards and Kapoor [13].

Tutte's famous characterization of regular matroids (see [20, 21]) has been generalized to delta-matroids arising from symmetric matrices [11]. The present work is motivated by the study of delta-matroids represented by skew-symmetric matrices, and the fact that proofs of Tutte's characterization of regular matroids rely heavily on Theorem 1.2. (See, for example, [14]).

2 Motivations and applications

The arguments in this paper are mainly graph theoretic, so we begin by restating the problem in terms of graphs. Throughout this paper all graphs will be assumed to be simple. The *adjacency matrix* of an undirected graph $G = (V, E)$ is the V by V symmetric $(0, 1)$-matrix that has a 1 in entry i, j if and only if $ij \in E$. The *adjacency matrix* of a directed graph $\vec{G} = (V, \vec{E})$ is the V by V skew-symmetric $(0, \pm 1)$-matrix that has a 1 in entry i, j if and only if $ij \in \vec{E}$. A digraph $\vec{G}$ is called an *orientation (PU-orientation)* of a graph G if the adjacency matrix of $\vec{G}$ is an orientation (PU-orientation) of the adjacency matrix of G. For an orientation $\vec{G}$ of G, we define the operations of *negation*, *cut-switching* and *switching* for $\vec{G}$ as the result of applying the corresponding operations to the adjacency matrix of $\vec{G}$.

Let $G = (V, E)$ be a graph, and let X, Y be disjoint subsets of V. We denote by $[X]$ the set of all distinct pairs of vertices in X, and we denote by $[X, Y]$ the set of all pairs of vertices containing an element of X and an element of Y. For $S \subseteq E$ we denote by $S[X]$ and $S[X, Y]$ the edge sets $S \cap [X]$ and $S \cap [X, Y]$ respectively. The set $E[X, Y]$ is referred to as a *cut* of G. The graph *induced* by X, denoted $G[X]$, is the graph $(X, E[X])$. For a vertex $v \in V$, we denote by $N_G(v)$ the neighbour set of v. For a graph G' we denote by $V_{G'}$ and $E_{G'}$ its vertex-set and edge-set.

A *split* of G is a partition (X, Y) of V such that $|X|, |Y| \geq 2$, and the cut $E[X, Y]$ induces a complete bipartite graph. (Note that not all pairs $x \in X, y \in Y$ need be joined by an edge for (X, Y) to be a split; in fact, if the cut $E[X, Y]$ contains no edges (X, Y) is a split.) A *prime* graph is a graph without any splits; thus, a graph is prime if and only if its adjacency matrix is prime.

Circle Graphs

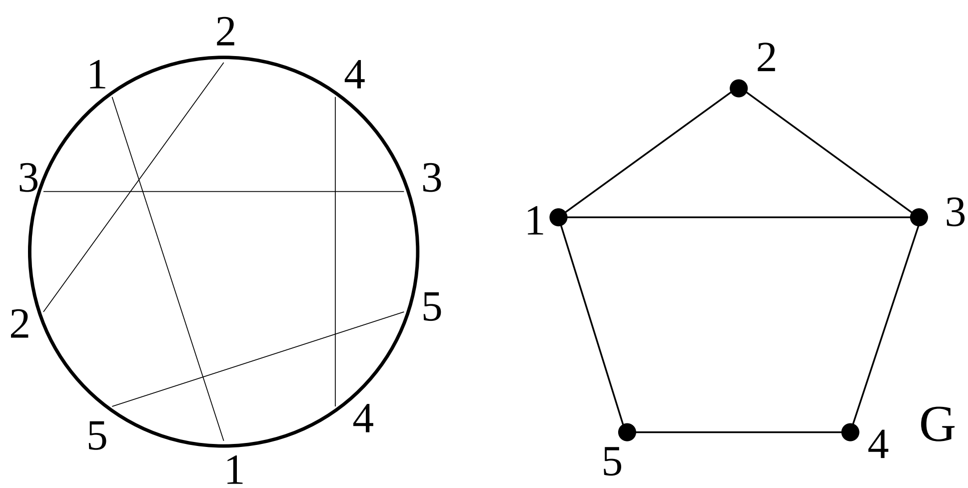

Figure 1: Circle graphs.

In this section we discuss an important class of graphs that admit PU-orientations, namely the circle graphs. A *circle graph* is the intersection graph of a finite set of chords of a circle. (See Figure 1.) De Fraysseix [10], showed that the bipartite circle graphs are the fundamental graphs of planar graphs. (If T is a spanning tree of a connected graph G then a *fundamental graph* of G is a bipartite graph with bipartition $E_T, E_G \setminus E_T$ and edges ef where $e \in E_T$, $f \in E_G \setminus E_T$ and $T + f - e$ is a tree.) It is well known that the fundamental matrices (that is, the adjacency matrices of fundamental graphs) of any graph can be signed to be totally unimodular. Hence bipartite circle graphs admit PU-orientations. In fact, this result extends to all circle graphs.

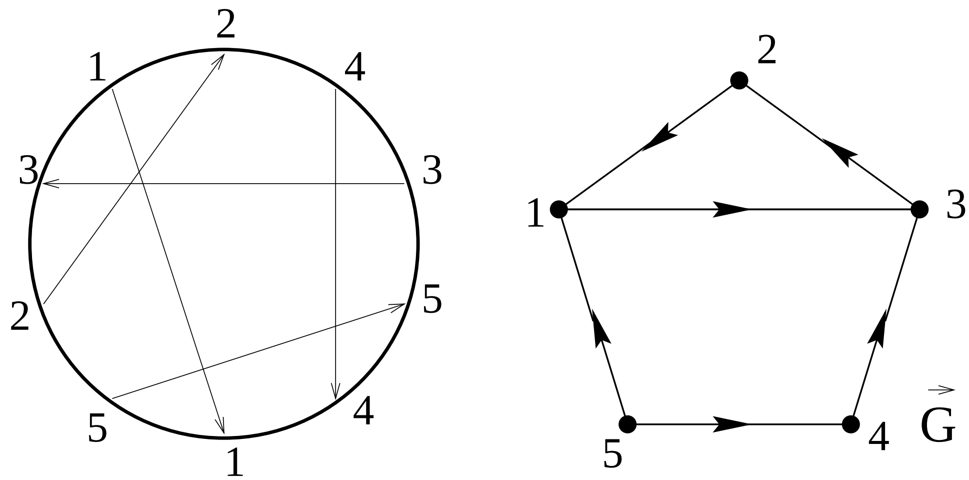

Figure 2: Orienting circle graphs.

Let $G = (V, E)$ be a circle graph represented by a set V of chords of a circle. By possibly perturbing the representation, we may assume that no two chords intersect on the circle. Given an arbitrary orientation of the chords, we define an orientation $\vec{G}$ of G. Namely, an edge uv of G is oriented with v as its head if and only if the chord v crosses u from left to right (that is, the tail of v is encountered before the head of u when the circle is traversed in the clockwise direction from the tail of u). Figure 2 depicts an arbitrary orientation of the representation in Figure 1 and the corresponding orientation of the circle graph. $\vec{G}$ is a PU-orientation of G. (see [3, 6].)

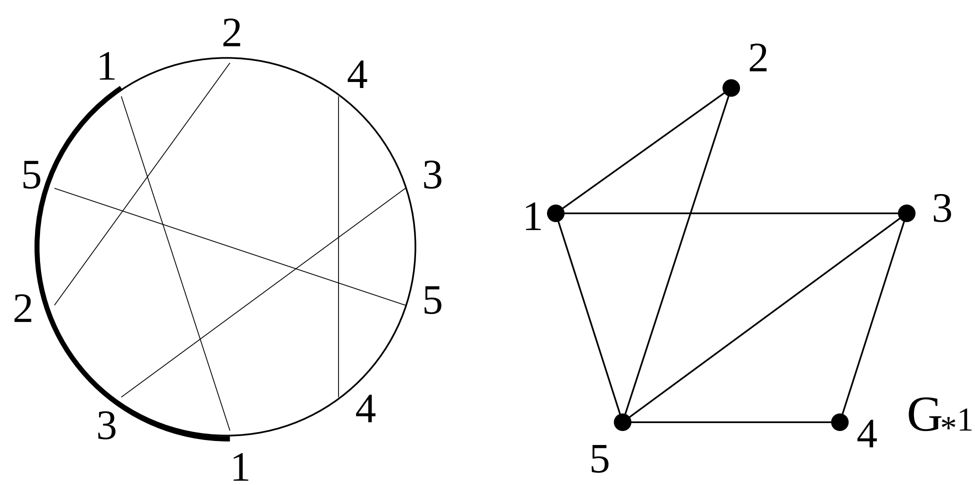

Figure 3: Local complementation.

Given a vertex v of a graph G, we define a new graph $G * v$ by complementing the induced graph on the neighbour set of v in G; this operation is called *local complementation.*

Kotzig [18] noted that G is a circle graph if and only if $G * v$ is a circle graph. Figure 3 demonstrates local complementation on the graph in Figure 1 and the new representation. (In general, if G is a circle graph, then a representation of $G * v$ can be obtained from a representation of G by reversing the order in which chords are encountered while traversing the circle in a clockwise direction from one end of v to the other.) Graphs that admit PU-orientations are not in general preserved under local complementation (in fact, G is a circle graph if and only if every graph equivalent to G under any sequence of local complementations admits a PU-orientation [7]). However, for an edge uv of a graph G, G admits a PU-orientation if and only if $G * u * v * u$ admits a PU-orientation; this operation is called a *pivot* and will be discussed further in Section 3.

Decomposition on splits

Let $G = (V, E)$ be a simple graph, and (X, Y) be a partition of V; denote by $G \circ X$ the graph obtained from G by shrinking X to a single vertex, which we label X, and removing multiple edges. Similarly if $\vec{G}$ is an orientation of G we define $\vec{G} \circ X$ by shrinking X in $\vec{G}$ and removing multiple arcs. Note that $\vec{G} \circ X$ need not be an orientation of $G \circ X$; however, if the edges in $E[X, Y]$ were all oriented in $\vec{G}$ with their heads in Y then $\vec{G} \circ X$ is an orientation of $G \circ X$. The decomposition $G \circ X$, $G \circ Y$ of G where (X, Y) is a split was studied in [9]; it has applications in Δ-matroid connectivity [5] and circle graph recognition [2]. Note that $G \circ X$ and $G \circ Y$ are both isomorphic to induced subgraphs of G; hence, if G has a PU-orientation then $G \circ X$ and $G \circ Y$ both have PU-orientations. In this section we show that the converse also holds, that is, if $G \circ X$ and $G \circ Y$ both admit PU-orientations then G admits a PU-orientation. Let $\vec{G}_1$ and $\vec{G}_2$ be PU-orientations of $G \circ X$ and $G \circ Y$ respectively. By cut-switching in $\vec{G}_1$, we may assume that no arc in $\vec{G}_1$ has X as its head. Similarly, we may assume that no arc in $\vec{G}_2$ has Y as its tail. Now construct an orientation $\vec{G}$ of G such that $\vec{G} \circ X = \vec{G}_1$ and $\vec{G} \circ Y = \vec{G}_2$; $\vec{G}$ is called the *composition* of $\vec{G}_1$ and $\vec{G}_2$. Before proving that $\vec{G}$ is a PU-orientation, we review some basic results about pfaffians; we use the definition of Stembridge [19].

Let $\vec{G}$ be an orientation of a simple graph $G = (V, E)$, let $A = (a_{ij})$ be the adjacency matrix of $\vec{G}$, let $\mathcal{M}_G$ denote the set of perfect matchings of G, and let $\prec$ be a linear order of V. A pair of edges u_1v_1, u_2v_2 of G, where $u_1 \prec v_1$ and $u_2 \prec v_2$, are said to *cross* if $u_1 \prec u_2 \prec v_1 \prec v_2$ or $u_2 \prec u_1 \prec v_2 \prec v_1$. (If we place u_1, u_2, v_1, v_2 on the perimeter of a circle, according to the linear order, then u_1u_2 crosses v_1v_2 if and only if the chords u_1u_2 and v_1v_2 cross.) The *sign* of a perfect matching M of G, denoted σ_M, is $(-1)^k$ where k is the number of pairs of crossing edges in M. The *pfaffian* of A, denoted $pf(A)$, is defined as follows:

$$pf(A) = \sum_{M \in \mathcal{M}_G} \sigma_M \prod_{\substack{uv \in M \\ u \prec v}} a_{uv}. \tag{1}$$

Surprisingly $pf(A)$ is independent of the order relation; this is reflected by the fundamental identity $\det(A) = pf(A)^2$. Like determinants, pfaffians can be calculated by "row expansion" [15]:

$$pf(A) = \sum_{k=2}^{n} (-1)^{k+1} a_{v_1 v_k} pf(A[V \setminus \{v_1, v_k\}]), \tag{2}$$

where $V = \{v_1, v_2, \ldots, v_n\}$ and $v_i \prec v_{i+1}$, for $i = 1, 2, \ldots, n-1$.

Proposition 2.1 *Let G be a graph containing a split (X, Y). Then the composition of PU-orientations of $G \circ X$ and $G \circ Y$ is a PU-orientation of G.*

Proof Let $\vec{G}_1$ and $\vec{G}_2$ be PU-orientations of $G \circ X$ and $G \circ Y$ respectively, and let $\vec{G}$ be the composition of $\vec{G}_1$ and $\vec{G}_2$. Let A, A_1 and A_2 be the adjacency matrices of $\vec{G}$, $\vec{G}_1$ and $\vec{G}_2$ respectively, and let $S \subseteq V$. We are required to prove that $\det(A[S]) \in \{0, 1\}$, or equivalently that $pf(A[S]) \in \{0, \pm 1\}$. If $|S \cap X| < 2$ or $|S \cap Y| < 2$ then $\vec{G}[S]$ is isomorphic to an induced subgraph of $\vec{G}_1$ or $\vec{G}_2$; hence $\det(A[S]) \in \{0, 1\}$. Now, suppose $|S \cap X| \geq 2$ and $|S \cap Y| \geq 2$; then $(X \cap S, Y \cap S)$ is a split in $G[S]$. We assume, without loss of generality, that $V = S$.

Suppose $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_l\}$. Define a linear order $\prec$ such that

$$x_k \prec x_{k-1} \prec \ldots \prec x_1 \prec y_1 \prec y_2 \prec \ldots \prec y_l.$$

Recall that, for $S \subseteq E_G$, $S[X, Y]$ denotes $S \cap [X, Y]$. Let $\mathcal{M}_G^{(i)} = \{M \in \mathcal{M}_G : |M[X, Y]| = i\}$; then, by (1),

$$pf(A) = \sum_{i \geq 0} \sum_{M \in \mathcal{M}_G^{(i)}} \sigma_M \prod_{\substack{uv \in M \\ u \prec v}} a_{uv}.$$

Claim 1 *For $i \geq 2$,*

$$\sum_{M \in \mathcal{M}_G^{(i)}} \sigma_M \prod_{\substack{uv \in M \\ u \prec v}} a_{uv} = 0.$$

Proof of claim: For each matching $M \in \mathcal{M}_G^{(i)}$, we define another matching M' as follows: choose edges $x_{i_1} y_{j_1}$ and $x_{i_2} y_{j_2}$, where $i_1 < i_2$, such that

$$M[\{x_1, x_2, \ldots, x_{i_2}\}, Y] = \{x_{i_1} y_{j_1}, x_{i_2} y_{j_2}\};$$

then define

$$M' = M \Delta \{x_{i_1} y_{j_1}, x_{i_2} y_{j_2}, x_{i_1} y_{j_2}, x_{i_2} y_{j_1}\}.$$

Note that $M = (M')'$, and

$$\sigma_M \prod_{\substack{uv \in M \\ u \prec v}} a_{uv} = -\sigma_{M'} \prod_{\substack{uv \in M' \\ u \prec v}} a_{uv},$$

which proves the claim.

For any perfect matching M of G we have $|M[X, Y]| \equiv |X| \pmod 2$; this gives rise to two cases.

Case 1: $|X|$ is even. Thus

$$\begin{aligned} pf(A) &= \sum_{M\in\mathcal{M}_G^{(0)}} \sigma_M \prod_{\substack{uv\in M\\ u\prec v}} a_{uv} \\ &= \sum_{M_X\in\mathcal{M}_G[X]} \sum_{M_Y\in\mathcal{M}_G[Y]} \sigma_{M_X\cup M_Y} \prod_{\substack{uv\in M_X\cup M_Y\\ u\prec v}} a_{uv} \\ &= \left(\sum_{M_X\in\mathcal{M}_G[X]} \sigma_{M_X} \prod_{\substack{uv\in M_X\\ u\prec v}} a_{uv} \right) \left(\sum_{M_X\in\mathcal{M}_G[X]} \sigma_{M_X} \prod_{\substack{uv\in M_X\\ u\prec v}} a_{uv} \right) \\ &= pf(A[X])pf(A[Y]). \end{aligned}$$

However $A[X]=A_2[X]$ and $A[Y]=A_1[Y]$, and A_1 and A_2 are PU, so $pf(A)=0,\pm 1$.
Case 2: $|X|$ is odd. Thus

$$pf(A) = \sum_{M\in\mathcal{M}_G^{(1)}} \sigma_M \prod_{\substack{uv\in M\\ u\prec v}} a_{uv}.$$

Every matching $M\in\mathcal{M}_G^{(1)}$ can be expressed as $M_1\cup M_2\cup\{x_iy_j\}$, where $M_1\in\mathcal{M}_{G[X-x_i]}$ and $M_2\in\mathcal{M}_{G[Y-y_i]}$. The set of edges of M that cross x_iy_j is

$$M_1[\{x_1,\ldots,x_{i-1}\},\{x_{i+1},\ldots,x_k\}]\cup M_2[\{y_1,\ldots,y_{j-1}\},\{y_{j+1},\ldots,y_l\}];$$

furthermore

$$\begin{aligned} |M_1[\{x_1,\ldots,x_{i-1}\},\{x_{i+1},\ldots,x_k\}]| &\equiv i-1 \pmod 2 \text{ and} \\ |M_2[\{y_1,\ldots,y_{j-1}\},\{y_{j+1},\ldots,y_l\}]| &\equiv j-1 \pmod 2. \end{aligned}$$

Therefore $\sigma_M=((-1)^{i-1}\sigma_{M_1})((-1)^{j-1}\sigma_{M_2})$, and

$$\begin{aligned} pf(A) &= \sum_{i=1}^{k}\sum_{j=1}^{l} \sum_{M_1\in\mathcal{M}_{G[X-x_i]}} \sum_{M_2\in\mathcal{M}_{G[Y-y_i]}} ((-1)^{i-1}\sigma_{M_1})((-1)^{j-1}\sigma_{M_2}) \\ &\qquad a_{x_iy_j} \left(\prod_{\substack{uv\in M_1\\ u\prec v}} a_{uv} \right) \left(\prod_{\substack{uv\in M_2\\ u\prec v}} a_{uv} \right) \\ &= \left(\sum_{i=1}^{k} (-1)^{i+1} \sum_{M_1\in\mathcal{M}_{G[X-x_i]}} \sigma_{M_1} \prod_{\substack{uv\in M_1\\ u\prec v}} a_{uv} \right) \\ &\qquad \left(\sum_{j=1}^{l} (-1)^{j+1} \sum_{M_2\in\mathcal{M}_{G[Y-y_i]}} \sigma_{M_2} \prod_{\substack{uv\in M_2\\ u\prec v}} a_{uv} \right). \end{aligned}$$

Now, applying equations (1) and (2),

$$\begin{aligned} pf(A) &= \left(\sum_{i=1}^{k} (-1)^{i+1} pf(A[X-x_i]) \right) \left(\sum_{j=1}^{l} (-1)^{j+1} pf(A[Y-y_i]) \right). \\ pf(A) &= -pf(A_1)pf(A_2), \end{aligned}$$

and hence, $pf(A) \in \{0, \pm 1\}$, as required. □

We remark that there is an alternative proof of Proposition 2.1 that uses pivoting (defined in section 3).

Counting PU-orientations

Let $G = (V, E)$ be a graph with a PU-orientation, and define $\alpha(G)$ to be the number of PU-orientations of G distinct up to cut-switching. Camion's theorem tells us that if G is bipartite then $\alpha(G) = 1$; the main result of this paper implies that if G is prime, but not bipartite, then $\alpha(G) = 2$. In this section we describe how $\alpha(G)$ can be computed by a canonical decomposition of graphs into graphs that are either prime, bipartite, or complete.

Let $\vec{G}$ be an orientation of G, and let C be an even circuit of G. We say that $\vec{G}$ is *even* (*odd*) on C if, while traversing C in an arbitrary direction, the number of edges of C that are oriented in the forward direction by $\vec{G}$ is even (odd). Because C has an even number of edges this definition is independent of the direction in which we traverse C.

Lemma 2.2 *Let C be the circuit x_1, x_2, x_3, x_4, x_1 of a graph G, and let $\vec{G}$ be a PU-orientation of G that is odd on C. Then $G[\{x_1, x_2, x_3, x_4\}]$ is a complete graph and $\vec{G}$ is even on the circuit x_1, x_2, x_4, x_3, x_1.*

Proof This follows by an easy pfaffian calculation, which is left to the reader. □

Let (X_1, X_2) and (Y_1, Y_2) be splits of G. We say that (X_1, X_2) and (Y_1, Y_2) *cross* if $X_i \cap Y_j \neq \emptyset$ for each i, j; we call the split (X_1, X_2) *good* if there are no splits of G that cross (X, Y). We recursively define a *decomposition* of a graph G as follows.

- $D = \{H : H \text{ a connected component of } G\}$ is a decomposition of G,
- If D is a decomposition of G and $H \in D$ has a good split (X, Y) then $(D \setminus H) \cup \{H \circ X, H \circ Y\}$ is a decomposition of G.

We call the elements of a decomposition D the *D-components*.

Theorem 2.3 *If D is a decomposition of G then $\alpha(G) = \prod_{H \in D} \alpha(H)$.*

Proof It is clear that $\alpha(G)$ is the product, taken over all connected components H of G, of $\alpha(H)$. Thus, it is sufficient to prove that if (X, Y) is a good split of G then $\alpha(G) = \alpha(G \circ X)\alpha(G \circ Y)$. By the composition of PU-orientations of $G \circ X$ and $G \circ Y$, we have that $\alpha(G) \geq \alpha(G \circ X)\alpha(G \circ Y)$. Therefore, it suffices to show that every PU-orientation $\vec{G}$ of G is equivalent under cut-switching to a composition of PU-orientations of $G \circ X$ and $G \circ Y$ (that is, $\vec{G}$ can be reoriented by cut-switching so that every edge in $E(X, Y)$ is oriented with its head in Y). Suppose, by way of contradiction, that $\vec{G}$ is a PU-orientation of G, and that $\vec{G}$ is not the composition of PU-orientations of $G \circ X$ and $G \circ Y$.

Let $X' = N_G(Y)$ and $Y' = N_G(X)$. Choose $x_1 \in X'$ and $y_1 \in Y'$; then, for all $y \in Y'$ and $x \in X'$, use cut-switching so that the edge $x_1 y$ is oriented with x_1 as the tail, and the edge $x y_1$ is oriented with y_1 as the head in $\vec{G}$. Since $\vec{G}$ is not the composition of PU-orientations of $G \circ X$ and $G \circ Y$, there exists an edge $x_2 y_2$ of G, where $x_2 \in X'$ and $y_2 \in Y'$, that is oriented with x_2 as its head. Partition X' into sets X_1, X_2 such that $x \in X_1$ if and only if the edge xy_2 has y_2 as its head; similarly, partition Y' into sets Y_1, Y_2 such that $y \in Y_1$ if and only if the edge x_2y has y as its head.

For any $y'_i \in Y_i$ $(i = 1, 2)$, $\vec{G}$ is odd on the circuit $x_1, y'_1, x_2, y'_2, x_1$, so, by Lemma 2.2, $G[\{x_1, x_2, y'_1, y'_2\}]$ is a complete graph. Therefore $y'_1y'_2$ is an edge of G. We similarly prove that $x'_1x'_2$ is an edge of G for any $x'_i \in X_i$ $(i = 1, 2)$. Hence $(X_1 \cup Y_1, X_2 \cup Y_2)$ is a split of $G[X_1 \cup X_2 \cup Y_1 \cup Y_2]$. However, since (X, Y) is a good split, there cannot exist a split (X'', Y'') with $X_1, Y_1 \subseteq X''$ and $X_2, Y_2 \subseteq Y''$. Therefore, there exists a chordless path $v_1, \ldots, v_p$ in $V \setminus (X' \cup Y')$ such that $N_G(v_i) \cap (X_1 \cup Y_1) \neq \emptyset$ if and only if $i = 1$, and $N_G(v_j) \cap (X_2 \cup Y_2) \neq \emptyset$ if and only if $j = p$. Since (X, Y) is a split in G, $\{v_1, \ldots, v_p\}$ is a subset of either X or Y; we assume, by possibly exchanging the roles of X and Y, that $\{v_1, \ldots, v_p\}$ is a subset of Y. Choose $y'_1 \in Y_1$ adjacent to v_1, and choose $y'_2 \in Y_2$ adjacent to v_p. Let H be the graph induced by $\{x_1, x_2, y'_1, y'_2, v_1, \ldots, v_p\}$; this is depicted by Figure 4.

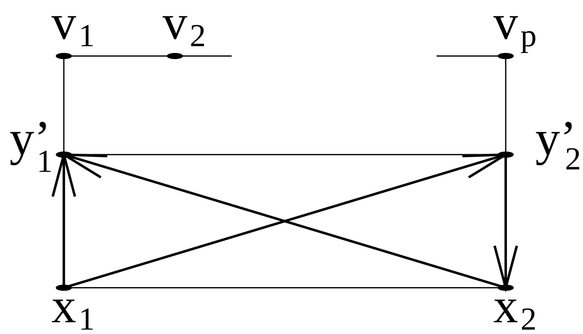

Figure 4: H

We assume that $p = 1$ or 2, since otherwise we shorten the path $y'_1, v_1, v_2, \ldots, v_p, y'_2$ by pivoting on v_1v_2, and then deleting v_1 and v_2 from G. If $p = 1$ then $\vec{G}$ is odd on exactly one of the circuits $v_1, y'_1, x_1, y'_2, v_1$ and $v_1, y'_1, x_2, y'_2, v_1$, which, by Lemma 2.2, contradicts that v_1 is adjacent to neither x_1 nor x_2. If $p = 2$ then pivoting on v_1v_2 deletes the edge $y'_1y'_2$ while leaving $\vec{G}$ odd on the circuit $x_1, y'_1, x_2, y'_2, x_1$, contradicting Lemma 2.2. □

Lemma 2.4 *For every integer n, $\alpha(K_n) = (n-1)!$, where K_n is the complete graph on n vertices.*

Proof Let $\vec{K}_n$ be a PU-orientation of K_n, and let v be any vertex of K_n. There exists a unique orientation equivalent under cut-switching to $\vec{K}_n$ with the property that every edge incident with v has v as its tail; we assume that $\vec{K}_n$ has this property.

Suppose that $\vec{K}_n$ has a directed circuit, and let $\vec{C}$ be a shortest directed circuit. $\vec{C}$ must have length 3, since otherwise there exists a chord e of $\vec{C}$ and $\vec{C} + e$ contains a directed circuit shorter than $\vec{C}$. Let X be the vertex set of $\vec{C}$. $\vec{K}_n$ is odd on every circuit of length 4 in $K_n[X + v]$, which contradicts Lemma 2.2. Hence $\vec{K}_n$ contains no directed circuits. We call such an orientation *transitive.*

There are $(n-1)!$ transitive orientations of $K_n - v$; thus, $\alpha(K_n) \leq (n-1)!$, with equality only if every transitive orientation of K_n is PU. Every two transitive orientations are isomorphic, so we may assume that $V_{K_n} = \{1, \ldots, n\}$, and for $1 \leq i < j \leq n$, the edge i, j is oriented with j as its head in $\vec{K}_n$. We have that $\vec{K}_3$ is PU; and, for $n > 3$, K_n is the composition of transitive orientations of two smaller complete graphs. Therefore, by Proposition 2.1 and induction, $\vec{K}_n$ is PU. □

A decomposition D is called a *total decomposition* if no D-component has a good split. A *star graph* with n vertices is a graph containing a vertex that is adjacent to $n-1$ vertices of degree 1. Total decompositions were introduced in [9], though our definition of the term *decomposition* differs slightly from the original definition.

Theorem 2.5 (Cunningham [9]) *Let G be a graph. Then*

- *All total decompositions of G are essentially the same; specifically, if D_1 and D_2 are total decompositions of G, then there exists a bijection $\pi : D_1 \to D_2$ such that, for each D_1-component H, H and $\phi(H)$ are isomorphic.*
- *If D is the total decomposition of G then every D-component is a complete graph, a star graph, or a prime graph.*
- *The total decomposition can be found in polynomial time.* □

Let D be the total decomposition of a graph G. By Theorem 2.5, every D-component H is either complete, prime or bipartite; so, assuming that G has a PU-orientation, we know $\alpha(H)$. Therefore, by Theorem 2.3, we know $\alpha(G)$ explicitly.

3 Prime graphs

This section contains the proof of Theorem 1.1. We begin by introducing preliminary results.

Pivoting

Let $A = (a_{ij})$ be a skew-symmetric $(0, \pm 1)$-matrix whose rows and columns are indexed by V. Suppose $u, w \in V$ and $a_{uw} = 1$. Define x, y so that

$$A = \left(\begin{array}{c|c|c} 0 & 1 & x^T \\ \hline -1 & 0 & y^T \\ \hline -x & -y & A[V-u-w] \end{array}\right)$$

where the first and second row are indexed by u and w respectively. Then define a matrix A' whose rows and columns are also indexed by the set V as follows:

$$A' = \left(\begin{array}{c|c|c} 0 & 1 & y^T \\ \hline -1 & 0 & x^T \\ \hline -y & -x & A[V-u-w] - yx^T + xy^T \end{array}\right).$$

The operation that constructs A' from A is called a *pivot* on uw in A. If in addition we switch the labels u and w, then we call the operation a *partial pivot*. The following result implies that the family of PU-matrices is closed under pivoting (and hence also under partial pivoting).

Proposition 3.1 *For $S \subseteq V$,* $\det A[S] = \det A'[S\Delta\{x,y\}]$.

Proof Since pivoting on uw in A has the same effect on principal submatrices of $A[S \cup \{u,w\}]$ as pivoting on uw in $A[S\cup\{u,w\}]$, we may assume that $S \cup \{u,w\} = V$. Furthermore, since pivoting is an involution, we may assume that $u \in S$. Hence it suffices to prove the following two identities:

$$\det A[V-w] = \det A'[V-u], \text{ and} \quad (3)$$
$$\det A = \det A'[V-u-w]. \quad (4)$$

Note that $A[V-w]$ and $A'[V-u]$ are equivalent under row and column operations. Thus, since the determinant is invariant under row and column operations, we have proved (3). Define

$$B = \left(\begin{array}{c|c|c} 0 & 1 & x^T \\ \hline -1 & 0 & y^T \\ \hline \underline{0} & \underline{0} & A[V-u-w] - yx^T + xy^T \end{array}\right).$$

B is obtained from A by row elimination, so $\det A = \det B$; furthermore

$$\det B = \det B[V-u-w] = \det A'[V-u-w].$$

Thus we have proved (4). □

For a pair S, S' of subsets of V, if S and S' are disjoint, we have defined $[S,S'] = \{ss' : s \in S, s' \in S'\}$; for intersecting sets S, S' we define

$$[S,S'] = [S \setminus S', S' \setminus S] \cup [S \setminus S', S \cap S'] \cup [S' \setminus S, S \cap S'].$$

We can interpret partial pivoting over the binary field as a transformation of an undirected graph. Let $G = (V,E)$ be the graph whose adjacency matrix is equivalent to A over $GF(2)$. Define a graph $G' = (V,E')$ where

$$E' = E\Delta[N_G(u) - w, N_G(w) - u].$$

It is easily verified that the adjacency matrix of G' is obtained by performing a partial pivot on uw in A over $GF(2)$.

A consequence of Proposition 3.1 is that pivoting (or partial pivoting) on a PU-matrix yields a $(0,\pm1)$-matrix. Thus we can think of pivoting and partial pivoting as operations on oriented graphs. Suppose A is PU and let $\vec{G} = (V,\vec{E})$ be the directed graph having adjacency matrix A. Let $\vec{G}' = (V,\vec{E}')$ be the directed graph whose adjacency matrix is obtained by performing a partial pivot on uw (over the reals) in A. Then we say that $\vec{G}'$ is obtained from $\vec{G}$ by performing a partial pivot on uw. Note that the orientation of uw is reversed by the partial pivot. The only other common edges of G and G' that may be oriented differently in $\vec{G}$ and $\vec{G}'$ are edges whose ends are both common neighbours of u and w.

The following result links pivoting and splits; in particular it implies that pivoting preserves prime graphs. It is implied by the fact that local complementation (defined in Section 2) preserves splits (see [2]) and that pivoting on an edge uw of G is equivalent to locally complementing on u, w, u in sequence.

Proposition 3.2 (Bouchet [5]) *Let (X, Y) be a partition of V, let $vw \in E$ and let $G' = (V, E')$ be the graph obtained by pivoting on vw in G. Then (X, Y) is a split in G if and only if (X, Y) is a split in G'.* □

Blocking sequences

A *subsplit* of G is a pair (X, Y) of disjoint subsets of V such that (X, Y) is a split in $G[X \cup Y]$ and the cut $E_G[X, Y]$ is nonempty. A **blocking sequence** for the subsplit (X, Y) is a sequence $v_1, \ldots, v_p$ of vertices in $V \setminus X \setminus Y$ satisfying the following conditions:

1. **(a)** $(X, Y \cup \{v_1\})$ is *not* a subsplit of G,
(b) for all $i < p$, $(X \cup \{v_i\}, Y \cup \{v_{i+1}\})$ is *not* a subsplit of G, and
(c) $(X \cup \{v_p\}, Y)$ is *not* a subsplit of G, and

2. no proper subsequence of $v_1, \ldots, v_p$ satisfies 1.

We note that the problem of finding a blocking sequence for (X, Y) can be solved by finding a directed path in the digraph $D = D(X, Y)$, with the vertex-set

$$V(D) = \{v_X, v_Y\} \cup (V \setminus X \setminus Y)$$

and the set of directed edges

$$\begin{aligned} E(D) = \; & \{(v_X, y) : (X, Y \cup \{y\}) \text{ is not a subsplit}\} \cup \\ & \{(x, y) : (X \cup \{x\}, Y \cup \{y\}) \text{ is not a subsplit}\} \cup \\ & \{(x, v_Y) : (X \cup \{x\}, Y) \text{ is not a subsplit}\}. \end{aligned}$$

Then $v_1, v_2, \cdots, v_p$ is a blocking sequence if and only if $v_X, v_1, v_2, \cdots, v_p, v_Y$ is a directed path with no shortcut in D. If no directed path exists in D, from v_X to v_Y, then the set

$$X'' = \{s \in V(D) - v_X : \text{ a directed path joins } v_X \text{ to } s\}$$

does not contain v_Y, and $(X', Y') := (X \cup X'', V \setminus X \setminus X'')$ is a subsplit of G.

Proposition 3.3 *Let (X, Y) be a subsplit of G. There exists a blocking sequence for (X, Y) in G if and only if there exists no split (X', Y') of G with $X \subseteq X'$ and $Y \subseteq Y'$.*
Proof If there exists a split (X', Y') of G with $X \subseteq X'$ and $Y \subseteq Y'$, then $(X \cup \{x\}, Y \cup \{y\})$ is a subsplit for every $x \in X' \setminus X$ and $y \in Y' \setminus Y$; therefore no blocking sequence exists. Conversely, if no blocking sequence exists, then we can find the required subsplit (X', Y') by using the digraph $D(X, Y)$. □

Following are some results that relate pivoting operations with blocking sequences.

Proposition 3.4 *Let (X, Y) be a subsplit of G and let G' be a graph obtained by performing a pivot (or partial pivot) on an edge of $G[X]$. A sequence $v_1, \ldots, v_p$ is a blocking sequence of (X, Y) in G if and only if it is a blocking sequence of (X, Y) in G'.*
Proof Let X', Y' be disjoint subsets of V with $X \subseteq X'$ and $Y \subseteq Y'$. By Proposition 3.2, (X', Y') is a subsplit of G' if and only if it is a subsplit of G. The result follows by considering the definition of a blocking sequence. □

Proposition 3.5 *Let $v_1, \ldots, v_p$ be a blocking sequence for a subsplit (X, Y) of G, let $x \in X \cap N_G(v_1)$ and let G' be the graph obtained by performing a partial pivot on the edge xv_1 in G. Suppose that $N_G(x) \cap X \neq \emptyset$ and $N_G(x) \cap X \neq N_G(Y) \cap X$. Then*

(i) if $p = 1$, (X, Y) is not a subsplit in G', and

(ii) if $p > 1$, $v_2, \ldots, v_p$ is a blocking sequence for (X, Y) in G'.

Proof **(i)** Suppose $p = 1$. Let $X' = N_G(Y) \cap X$ and $Y' = N_G(X) \cap Y$. Then, since (X, Y) is a subsplit, $E_G[X, Y] = [X', Y']$. Note that

$$[P, Q] \cap [R, S] = [P \cap R, Q \cap S] \Delta [P \cap R, Q \cap S]$$

holds for any subsets P, Q, R and S of V. Therefore

$$\begin{aligned} E_{G'}[X, Y] &= (E_G \Delta [N_G(v_1) - x, N_G(x) - v_1]) \cap [X, Y] \\ &= [X', Y'] \Delta [(N_G(v_1) - x) \cap X, N_G(x) \cap Y] \Delta [N_G(x) \cap X, N_G(v_1) \cap Y]. \end{aligned}$$

We consider two cases; in each case we use the following fact:

> Suppose $E_{G'}[X, Y] = [X_1, Y_1] \Delta [X_2, Y_2]$ where X_1 and X_2 are distinct nonempty subsets of X, and Y_1 and Y_2 are distinct, nonempty subsets of Y. Then (X, Y) is not a subsplit in G'.

Case 1: $x \notin X'$. Then $N_G(x) \cap Y = \emptyset$, so

$$E_{G'}(X, Y) = [X', Y'] \Delta [N_G(x) \cap X, N_G(v_1) \cap Y].$$

Furthermore, by the conditions of the proposition, X', $N_G(x) \cap X$ are distinct, nonempty subsets of X, and, by the definition of a blocking sequence, Y', $N_G(v_1) \cap Y$ are distinct, nonempty subsets of Y, so (X, Y) is not a subsplit in G'.

Case 2: $x \in X'$. Then $N_G(x) \cap Y = Y'$. Note that, for any sets $A \subseteq Y$, $B_1, B_2 \subseteq X$, $[A, B_1] \Delta [A, B_2] = [A, B_1 \Delta B_2]$, so

$$E_{G'}[X, Y] = [X' \Delta ((N_G(v_1) - x) \cap X), Y'] \Delta [N_G(x) \cap X, N_G(v_1) \cap Y].$$

Now $x \in X' \Delta ((N_G(v_1) - x) \cap X)$. However $x \notin N_G(x) \cap X$, so $X' \Delta ((N_G(v_1) - x) \cap X)$, $N_G(x) \cap X$ are distinct, nonempty subsets of X. Furthermore, by the definition of a blocking sequence, Y', $N_G(v_1) \cap Y$ are distinct nonempty subsets of Y; hence (X, Y) is not a subsplit in G'.

(ii) Suppose $p > 1$. By the minimality of a blocking sequence we have that $(X, Y \cup \{v_2\})$ is a subsplit in G. Note that v_1 is a blocking sequence for the subsplit $(X, Y \cup \{v_2\})$ in G. By part (i) of the proposition, $(X, Y \cup \{v_2\})$ is not a subsplit in G'. Also note that $(X \cup \{v_1\}, Y)$ is a subsplit in G and that $v_2, \ldots, v_p$ is a blocking sequence for $(X \cup \{v_1\}, Y)$ in G. By Proposition 3.4, $v_2, \ldots, v_p$ is also a blocking sequence for $(X \cup \{v_1\}, Y)$ in G', and, since $(X, Y \cup \{v_2\})$ is not a subsplit in G', $v_2, \ldots, v_p$ is also a blocking sequence for (X, Y) in G'. □

Sign-fixed circuits

Let C be a circuit in a graph G. We say that C is **sign-fixed** with respect to G if any two PU-orientations of G differ on an even number of edges of C. For subgraphs H_1, H_2 of G, we denote by $H_1 \Delta H_2$ the subgraph of G induced by the edges $E_{H_1} \Delta E_{H_2}$.

Proposition 3.6 *Let C be a circuit of a graph G. If there exist sign-fixed circuits $C_1, \ldots, C_k$ of G such that $C = C_1 \Delta C_2 \Delta \ldots \Delta C_k$ then C is sign-fixed in G.*

Proof Let $\vec{G}_1, \vec{G}_2$ be any pair of PU-orientations of G. Let S be the set of edges of G in which the orientations $\vec{G}_1$ and $\vec{G}_2$ differ. For each sign-fixed circuit C_i, $|C_i \cap S|$ is even. Now

$$\begin{aligned} C \cap S &= (C_1 \Delta \ldots \Delta C_k) \cap S \\ &= (C_1 \cap S) \Delta \ldots \Delta (C_k \cap S). \end{aligned}$$

Since $C \cap S$ can be represented as the symmetric difference of even sets, $C \cap S$ has even cardinality. Hence C is sign-fixed in G. □

The following proposition is attributed to Bondy in [16]; it can be proved using Menger's theorem.

Proposition 3.7 *Let H be an eulerian subgraph of a 2-vertex-connected graph G. If H has an even number of edges, then there exist even circuits $C_1, \ldots, C_k$ of G such that*

$$H = C_1 \Delta C_2 \Delta \ldots \Delta C_k.$$ □

Lemma 3.8 *Let G be a graph such that every even circuit is sign-fixed. All PU-orientations of G are switching-equivalent if G is bipartite or 2-connected.*

Proof Let $\vec{G}_1, \vec{G}_2$ be PU-orientations of G. If C' is an even circuit of G, then $\vec{G}_1$ and $\vec{G}_2$ differ on an even number of edges of C', by the premises of the lemma. We claim that the same property may be assumed for every circuit C' of G. This is obvious if G is bipartite. Otherwise fix an odd circuit C. We may assume that the orientations $\vec{G}_1$ and $\vec{G}_2$ differ on an even number of edges of C; otherwise we reverse the orientation $\vec{G}_2$. Consider any other odd circuit C' of G. By Proposition 3.7 there exist even circuits $C_1, \ldots, C_k$ such that $C' \Delta C = C_1 \Delta \ldots \Delta C_k$, therefore $C' = C \Delta C_1 \Delta \ldots \Delta C_k$. It follows similarly to the the proof of Proposition 3.6, that the orientations $\vec{G}_1$ and $\vec{G}_2$ differ on an even number of edges of C'. Which proves the claim.

Let S be the set of edges upon which the orientations $\vec{G}_1$ and $\vec{G}_2$ differ. It follows from the claim that if we contract each of the edges in $E_G \setminus S$, then we obtain a bipartite graph. Therefore the edges S form a cut in G, so $\vec{G}_1$ and $\vec{G}_2$ are equivalent under cut-switching. □

Corollary 3.9 *If G is prime and every even circuit of G is sign-fixed, then all PU-orientations of G are switching equivalent.*

Proof Trivially we may assume G has at least 4 vertices. Then G is 2-connected. □

Lemma 3.8 generalizes the ideas used in Seymour's proof of Theorem 1.2. Following is a summary of Seymour's proof. Suppose C is a circuit of a bipartite graph G. If C is chordless then it is easy to show that C is sign-fixed. Otherwise, if C has a chord, then C can be expressed as the symmetric difference of two shorter circuits, so inductively we can prove that C is sign-fixed. Then, by Lemma 3.8, all PU-orientations of G are switching-equivalent.

Decomposition of circuits

In this section we describe three decompositions of an even circuit C into a symmetric difference of shorter even circuits.

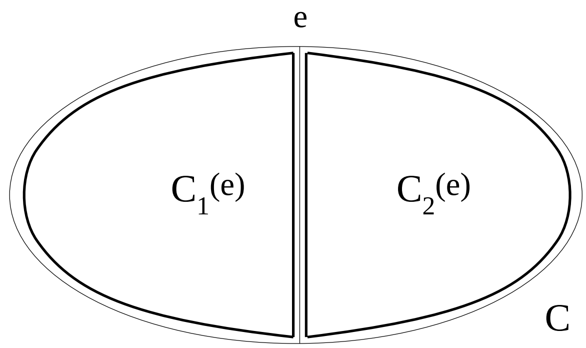

Figure 5: $C + e$

Let C be an even circuit and let e be a chord of C. C can be expressed as the symmetric difference of two shorter circuits (see Figure 5) denoted $C_1(e), C_2(e)$ (in no particular order). Since C is even, $C_1(e)$ and $C_2(e)$ are either both even or both odd. We say that e is an *even* (*odd*) chord of C if $C_1(e)$ and $C_2(e)$ are both even (odd). The first decomposition of C is $C = C_1(e)\Delta C_2(e)$, when e is an even chord.

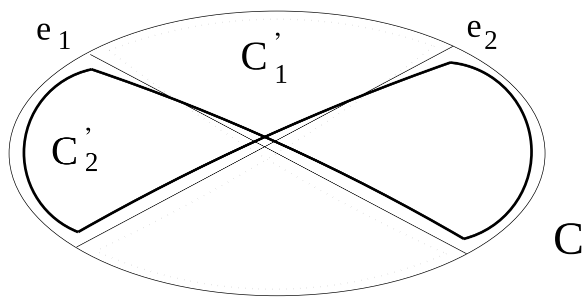

Figure 6: Decomposition of $C + e_1 + e_2$

Let e_1 and e_2 be odd chords of an even circuit C. We say that e_1 and e_2 *cross* if e_1 and e_2 have disjoint ends and e_2 has exactly one end in $C_1(e_1)$. If e_1 and e_2 are crossing then define $C_1' = C_1(e_1)\Delta C_1(e_2)$ and $C_2' = C_1(e_1)\Delta C_2(e_2)$; see Figure 6. C_1' and C_2' are both even circuits and

$$\begin{aligned} C_1'\Delta C_2' &= (C_1(e_1)\Delta C_1(e_1))\Delta(C_1(e_1)\Delta C_2(e_2)) \\ &= C_1(e_2)\Delta C_2(e_2) \\ &= C. \end{aligned}$$

If either C_1' or C_2' has length 4 then the other has the same length as C; otherwise both C_1' and C_2' are shorter than C. We say that e_1 and e_2 are *tight* crossing chords if either C_1' or C_2' has length 4. The second decomposition of C is $C = C_1'\Delta C_2'$, when e_1 and e_2 are not tight crossing chords.

Note that it is not possible to have three odd chords of a circuit such that each pair is a tight crossing pair, so if we have any three mutually crossing odd chords of a circuit C, we can apply one of the above decompositions to express C as the symmetric difference of two shorter even circuits.

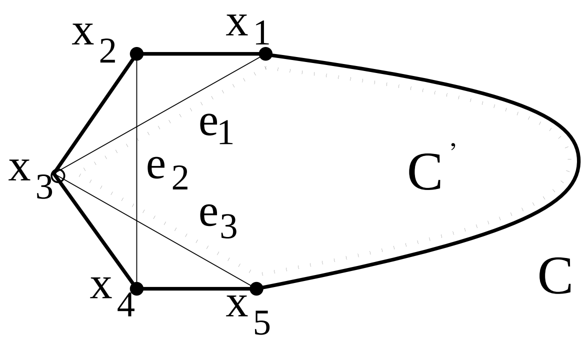

Figure 7: $C + e_1 + e_2 + e_3$

In the third decomposition we have three odd chords e_1, e_2 and e_3 of an even circuit C such that $\{e_1, e_2\}$ and $\{e_2, e_3\}$ are pairs of tight crossing chords and e_1 and e_3 do not cross. In this situation there are consecutive vertices $x_1, \ldots, x_5$ in C such that e_1, e_2 and e_3 have ends $\{x_1, x_3\}$, $\{x_2, x_4\}$ and $\{x_3, x_5\}$ respectively, as depicted in Figure 7. Also depicted in Figure 7 is an even circuit C'; C is the symmetric difference of C' and the two circuits x_1, x_2, x_4, x_3, x_1 and x_5, x_4, x_2, x_3, x_5. Furthermore each of these circuits is even and shorter than C.

A circuit is said to be *decomposable* (otherwise *indecomposable*) if by one of the above decompositions we can express C as the symmetric difference of shorter even circuits. More rigorously, an even circuit C is indecomposable if the chords of C are all odd, each chord crosses at most one other chord and all crossings are tight.

PU-orientations of prime graphs

We now prove the main result of the paper.

Proof of Theorem 1.1. By Corollary 3.9, it suffices to show that in a prime graph all even circuits are sign-fixed. We prove this by induction on the length of an even circuit. Let $k \geq 4$ be an even integer. We assume that in every prime graph every even circuit of length less than k is sign-fixed.

Let C' be a circuit of length k in a prime graph G'. If C' can be expressed as the symmetric difference of sign-fixed circuits in G' then, by Proposition 3.6, C' is sign-fixed. In particular, if C' is decomposable then C' is sign-fixed.

Claim 1 *Let C be a circuit of length k in a prime graph G. If there exists a vertex that has degree 2 in $G[V_C]$ then C is sign-fixed.*

Proof of claim In the case that C has length 4, the claim follows from Lemma 2.2. Now suppose that $k > 4$ and that C is indecomposable. Let v be a vertex of degree 2 in $G[V_C]$, let u, w be the neighbours of v in $G[V_C]$ and let G' be the graph obtained by performing a partial pivot on vw in G.

Let $u'u$ and ww' be the edges other than uv and uw incident to u and w respectively in C. Note that u' is not adjacent to w in G since such an edge would be an even chord of C, and similarly u is not adjacent to w'. We have that $N_{G[V_C]}(v) - w = \{u\}$, so

$$E_{G'}[V_C] = E_G[V_C]\Delta[\{u\}, N_{G[V_C]}(w) - v].$$

Therefore the partial pivot affects only edges incident with u, but the edges uu' and uv are unaffected by the partial pivot, so C is a circuit in G'. Furthermore if the partial pivot were performed on any orientation of G, then exactly one edge of C, namely vw, will be reoriented, so C is sign-fixed in G if and only if C is sign-fixed in G'. Now uw' is an edge of G', so C has an even chord in G'. Hence C is sign-fixed in G'. This proves Claim 1.

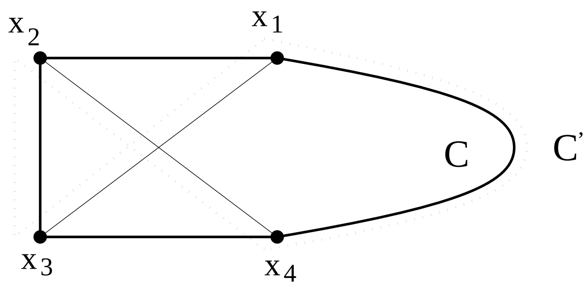

Figure 8: Circuits in Claim 2.

Claim 2 *Let C be a circuit of length k in a prime graph G, and suppose $x_1, \ldots, x_4$ are consecutive vertices of C such that x_1x_3 and x_2x_4 are chords of C. Finally let C' be the symmetric difference of C and the circuit x_1, x_3, x_4, x_2, x_1. (See Figure 8.) Then at least one of C and C' is sign-fixed.*

Proof of claim The claim is trivially true when C is decomposable, so suppose that C is indecomposable. Let $X = \{x_2, x_3\}$ and $Y = V_C \setminus X$, and let e_1 and e_2 be the edges x_1x_3 and x_2x_4, respectively. Note that e_1 and e_2 are crossing chords of C, so there are no other chords which cross either e_1 or e_2. Hence (X, Y) is a subsplit of G; let $v_1, \ldots, v_p$ be a blocking sequence for this subsplit. We prove the claim by induction on the length of the blocking sequence.

Case 1: $p = 1$. v_1 is a blocking sequence for the subsplit (X, Y) in G. Then v_1 is adjacent to exactly one of x_2 and x_3. Assume with no loss of generality that v_1 is adjacent to x_2. v_1 must also be adjacent to some vertex in Y. This gives rise to two subcases.

Case 1.1: v_1 is adjacent to a vertex y in $Y \setminus \{x_1, x_4\}$. We assume that x_2 and y are an even distance apart in C. (Otherwise x_2 and y are an even distance apart in C' and we can interchange the roles of C and C'.) Consider the circuits C_1 and C_2 defined by Figure 9. C_1 and C_2 are both even and have length at most k. x_3 and x_2 have degree 2 in $G[V_{C_1}]$ and $G[V_{C_2}]$ respectively, so by Claim 1, C_1 and C_2 are both sign-fixed. Furthermore C is the symmetric difference of C_1 and C_2 so C is also sign-fixed. This completes the proof of Claim 2 in Case 1.1.

Case 1.2: v_1 is not adjacent to any vertices in $Y \setminus \{x_1, x_4\}$. In this case v_1 cannot be adjacent to both x_1 and x_4 since otherwise $(X \cup \{v_1\}, Y)$ would be a subsplit, contradicting the definition of a blocking sequence. So v_1 is adjacent to exactly one of x_1 and x_4. We assume that v_1 is adjacent to x_1. (The other case is equivalent under interchanging the

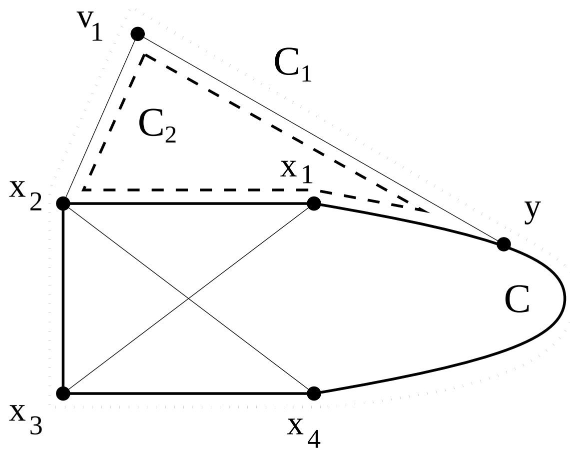

Figure 9: Decomposition in Case 1.1.

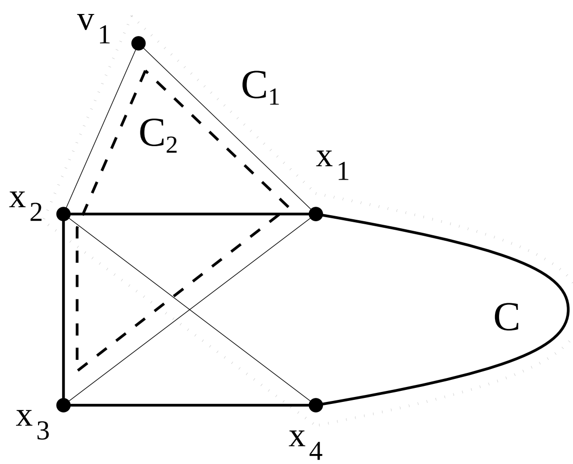

Figure 10: Decomposition in Case 1.2.

roles of C and C' and changing labels.) Consider the even circuits C_1 and C_2 defined by Figure 10. v_1 has degree 2 in both $G[V_{C_1}]$ and $G[V_{C_2}]$, so by Claim 1, C_1 and C_2 are both sign-fixed. C' is the symmetric difference of C_1 and C_2 so C' is also sign-fixed. This completes the proof of Claim 2 in Case 1.

Case 2: $p > 1$. As with Case 1, v_1 is adjacent to exactly one of x_2 and x_3, and we assume with no loss of generality that x_2 and v_1 are adjacent. $(X \cup \{v_1\}, Y)$ is a subsplit, so either $N_G(v_1) \cap Y = \emptyset$ or $N_G(v_1) \cap Y = N_G(X) \cap Y = \{x_1, x_4\}$. This gives two subcases.

Case 2.1: $N_G(v_1) \cap Y = \emptyset$. Let G' be the graph defined by performing a partial pivot on the edge x_2v_1. Note that $N_G(v_1) \cap V_C = \{x_2\}$, so $G[V_C] = G'[V_C]$. Then C and C' are circuits in G' and, by considering the effect of this partial pivot on an orientation of G, C and C' are sign-fixed in G if and only if they are sign-fixed in G'. Now, by Proposition 3.5, $v_2, \ldots, v_p$ is a blocking sequence for the subsplit (X, Y) in G', so, by the induction hypothesis of the claim, one of C and C' is sign-fixed in G'.

Case 2.2: $N_G(v_1) \cap Y = \{x_1, x_4\}$. We have that $v_2, \ldots, v_p$ is a blocking sequence for the subsplit $(X \cup \{v_1\}, Y)$. Furthermore, for $i > 1$, $(X, Y \cup \{v_i\})$ is a subsplit; it follows

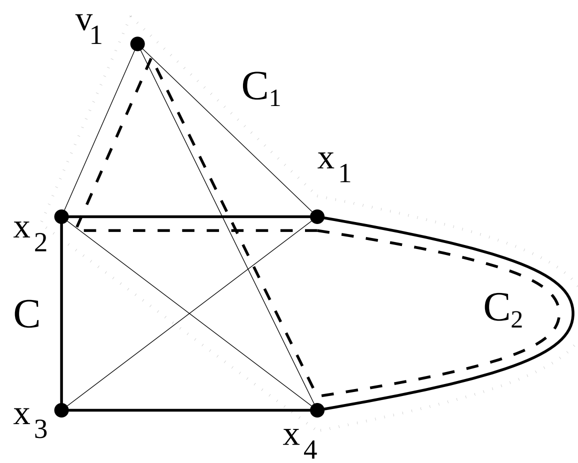

Figure 11: Decomposition in Case 2.2.

that v_i is adjacent with x_2 if and only if v_i is adjacent with x_3. Consequently $v_2, \ldots, v_p$ is a blocking sequence for the subsplit $(\{x_2, v_1\}, Y)$. Now, by the induction hypothesis of the claim, one of the circuits C_1 or C_2, defined in Figure 11, is sign-fixed. Let C_1' and C_2' be the circuits v_1, x_1, x_3, x_2, v_1 and v_1, x_4, x_3, x_2, v_1 respectively. C_1' and C_2' are both sign-fixed by Claim 1. If C_1 is sign-fixed then C', which is the symmetric difference of C_1 and C_1', is sign-fixed. Otherwise C_2 is sign-fixed; then C, which is the symmetric difference of C_2 and C_2', is sign-fixed. In either case we have proved Claim 2.

The proof is now settled with two final cases.

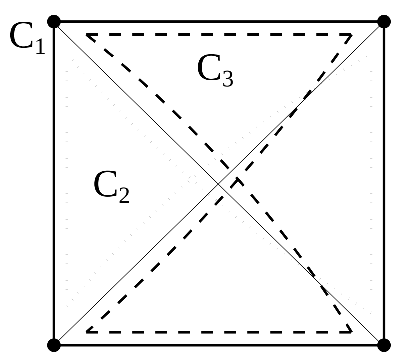

Figure 12: Decomposition when $k = 4$.

Case 1: $k = 4$. Let C_1 be a circuit of length 4 in a prime graph G. If $G[V_{C_1}]$ is not complete then $G[V_{C_1}]$ contains a vertex of degree 2, so, by Claim 1, C_1 is sign-fixed. Thus we may suppose that $G[V_{C_1}]$ is complete. Let C_2 and C_3 be defined by Figure 12. By Claim 2, one of C_1 and C_2 is sign-fixed. If C_1 is sign-fixed we are done, so suppose C_2 is sign-fixed. Similarly one of C_1 and C_3 are sign-fixed, so suppose C_3 is sign-fixed. However C_1 is the symmetric difference of C_2 and C_3, so C_1 is sign-fixed.

Case 2: $k > 4$. Let C be a circuit of length k in a prime graph G. If C is decomposable or if $G[V_C]$ contains a vertex of degree 2 then C is sign-fixed. Suppose then that C is indecomposable and that every vertex in $G[V_C]$ has degree at least 3. Let e be a chord of C such that the distance in C between the ends of e is minimum among all chords of

C. Let $y_1, \ldots, y_r$ be the internal vertices of a shortest path in C between the ends of e. Since each vertex in V_C has degree at least 3 in $G[V_C]$, each y_i must subtend at least one chord of C; let e_i be a chord having y_i as an end. The distance in C between the ends of e_i is at least the distance between the ends of e in C, so e_i must cross e. Since C is indecomposable, there is at most one chord crossing e; therefore $r = 1$. Furthermore e_1 and e must be a tight crossing pair, so the other end of e_1 must also be adjacent to an end of e in C. Therefore there are consecutive vertices x_1, x_2, x_3, x_4 of C such that x_1 and x_3 are the ends of e, and x_2 and x_4 are the ends of e_1. Let C' be the circuit x_1, x_2, x_4, x_3, x_1; C' is sign-fixed since it has length 4. By Claim 2 at least one of C and $C \Delta C'$ is sign-fixed. If C is sign-fixed we are done. Otherwise $C \Delta C'$ is sign-fixed, so C (which is the symmetric difference of $C \Delta C'$ and C') is also sign-fixed. This completes the proof. $\square$

4 Constructing a PU-orientation

Let $G = (V, E)$ be a simple graph that admits a PU-orientation. In Section 2 we essentially described how to construct all PU-orientations of G from a single PU-orientation. In this section, we outline a polynomial-time algorithm that provides the initial PU-orientation. By Proposition 2.1, we may assume that G is prime.

We fix an arbitrary orientation $\vec{G}_0 = (V, \vec{E}_0)$ of G. Thus orientations can be conveniently encoded by $(0,1)$-vectors indexed by E. Specifically, an orientation $\vec{G}$ is encoded by $x \in \{0,1\}^E$ where $x_e = 0$ if and only if $\vec{G}$ and $\vec{G}_0$ concur in their orientation of e. Henceforth we refer to an orientation by its encoding.

Let $\mathcal{C}_0$ denote the set of edge sets of even circuits of G. Let M be the incidence matrix of even circuits versus edges of G. That is, M is a $(0,1)$-matrix with rows $\mathcal{C}_0$ and columns E where, for $C \in \mathcal{C}_0$ and $e \in E$, the (C, e) entry of M is 1 if and only if $e \in C$. Let $v, v^* \in \{0,1\}^E$, where v^* is a PU-orientation, and let $b = Mv^*$. Then, by Theorem 1.1, v is PU if and only if v satisfies the binary matrix equation $Mv = b$. Let $\mathcal{B}_0 \subseteq \mathcal{C}_0$ be a basis of the even-circuit space (that is, the rowspace of M over GF(2)). We now define, $M' = M[\mathcal{B}_0, E]$ and $b' = M'v^*$. Then, $M'v = b'$ if and only if $Mv = b$. Consequently, *for $v \in \{0,1\}^E$, v is PU if and only if $M'v = b'$ over $GF(2)$.* Our algorithm finds a PU-orientation by solving the binary matrix equation $M'v = b'$. At this point there remain two obstacles in implementing the algorithm, namely:

(1) How can we find a basis for the even-circuit space efficiently?

(2) For an even circuit C, how can we compute b_C efficiently (without knowing v^*)?

Let $\mathcal{C}$ denote the set of edge sets of circuits of G. The circuit space (that is the rowspace, over $GF(2)$, of the circuit-edge incidence matrix of G) is the set of incidence vectors of eulerian subgraphs of G. Thus, by Proposition 3.7, there exists a basis $\mathcal{B} \subseteq \mathcal{C}$ of the circuit space that contains at most one odd circuit. For bipartite graphs this is trivial; for nonbipartite graphs such a basis can be constructed efficiently by making an ear decomposition of G that begins with an odd circuit; we leave the details to the reader. Given such a basis of the circuit space, the even circuits form a basis of the even-circuit space. This answers (1).

The second of the aforementioned problems is less trivial. However, our proof of Theorem 1.1 is essentially a recursive algorithm for computing b_C. The algorithm relies on the following strengthening of Proposition 3.6, whose proof is left to the reader.

Proposition 4.1 *Let $C, C_1, \ldots, C_k$ be even circuits of G such that $C = C_1 \Delta \ldots \Delta C_k$. Then $b_C = b_{C_1} + \ldots + b_{C_k}$ modulo 2.* □

Our algorithm immediately separates the cases where $|C| = 4$ and $|C| > 4$. However, in each case we must solve the subproblem given in Claim 2; precisely, the problem is as follows.

Subproblem: Let C be an even circuit with consecutive vertices $x_1, \ldots, x_4$ such that x_1x_3 and x_2x_4 are chords, and let C' be the symmetric difference of C and the circuit x_1, x_3, x_4, x_2, x_1. Find b_C or $b_{C'}$.

The algorithm for this subproblem comes directly from the proof of Claim 2. We leave the details to the reader, and instead focus on the main algorithm.

Suppose that $|C| = 4$. If $G[V_C]$ has a vertex of degree 2, then b_C can easily be computed using Lemma 2.2. Thus we assume $G[V_C]$ is complete, and is depicted in Figure 12. By using the subproblem twice, we determine two of $b_{C_1}, b_{C_2}, b_{C_3}$, and the third is obtained by their sum.

We now consider the case that $|C| > 4$.

If C is decomposable, then we can express C as the symmetric difference of circuits $C_1, \ldots, C_k$, as described in Figures 5, 6 and 7, such that $|C_i| < |C|$, for $i = 1, \ldots, k$, and $\sum_{i=1}^{k} |C_i| \leq |C| + 8$. Thus b_C can be computed recursively as the sum of the b_{C_i}. The conditions on the sizes of these circuits maintains the efficiency of the algorithm. Henceforth we assume that C is indecomposable.

Now suppose that $G[V_C]$ has a vertex v of degree 2. Let w be a vertex adjacent to v. Note that changing the orientation of an edge in $\vec{G}_0$ has a predictable effect on b_C. We change the orientation $\vec{G}_0$ so that we have the following property: *Each edge xy with $x \in N(w)$ and $y \in N(v)$ is oriented with its head being a neighbour of v and its tail being a neighbour of w.* (Note that we allow $x = v$ and $y = w$.) We leave it to the reader to check that this property ensures that partial pivoting on $\{v, w\}$ in the adjacency matrix of $\vec{G}_0$ yields a $(0, \pm 1)$-matrix. Let $\vec{G}'_0$ be the oriented graph obtained by this partial pivot, and let G' be the graph obtained by performing a partial pivot on vw in G. Note that $\vec{G}'_0$ is an orientation of G'; also C is a circuit of G' and b_C is unaffected by the pivot. However, the partial pivot added an even chord to C, making C decomposable. Henceforth we may assume that $G[V_C]$ has no vertex of degree 2.

By the assumptions on C, we can find consecutive vertices x_1, x_2, x_3, x_4 of C such that x_1x_3 and x_2, x_4 are chords. Let C_0 denote the circuit x_1, x_2, x_4, x_3, x_1. Since C is indecomposable, x_1x_4 is not a chord. Thus b_{C_0} can be computed easily by Lemma 2.2. Let C' be the symmetric difference of C and C_0. We now use the subproblem to find b_C or $b_{C'}$. Thus we know two of $b_{C_0}, b_C, b_{C'}$, their sum gives us the third. This completes the algorithm.

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References

[1] A. Bouchet. Greedy algorithm and symmetric matroids. *Math. Programming*, 38:147–159, 1987.

[2] A. Bouchet. Reducing prime graphs and recognizing circle graphs. *Combinatorica*, 7:243–254, 1987.

[3] A. Bouchet. Unimodularity and circle graphs. *Discrete Math.*, 66:203–208, 1987.

[4] A. Bouchet. Representability of Δ–matroids. *Colloquia Societatis Janos Bolyia*, 52:167–182, 1988.

[5] A. Bouchet. Matroid connectivity and fundamental graphs. In *Proceedings of the Twenty–Second Southeastern Conference on Combinatorics, Graph Theory and Computing*, 1991.

[6] A. Bouchet. A characterization of unimodular orientations of simple graphs. *J. Combin. Theory Ser. B*, 56:45–54, 1992.

[7] A. Bouchet. Circle graph obstructions. *J. Combin. Theory Ser. B*, 60:107–144, 1994.

[8] P. Camion. Caractérisation des matrices unimodulaires. *Cahiers Centre Études Rech. Opér.*, 5:181–190, 1963.

[9] W. H. Cunningham. Decomposition of directed graphs. *SIAM J. Algebra Discrete Math.*, 32:214–228, 1982.

[10] H. de Fraysseix. A characterization of circle graphs. *European J. Combin.*, 5:223–238, 1984.

[11] J. F. Geelen. A generalization of Tutte's characterization of totally unimodular matrices. In preparation.

[12] J. F. Geelen. *Matchings, matroids, and unimodular matrices.* PhD thesis, University of Waterloo, 1995.

[13] J. F. Geelen, A. M. H. Gerards, and A. Kapoor. The excluded minor characterization of GF(4)–representable matroids. In preparation.

[14] A. M. H. Gerards. A short proof of Tutte's characterization of totally unimodular matrices. *Linear Algebra Appl.*, 114/115:207–212, 1989.

[15] C. D. Godsil. *Algebraic Combinatorics.* Chapman and Hall, 1993.

[16] I. B.-A. Hartman. The Cycle Space of a Graph. Master's thesis, University of Waterloo, 1982.

[17] J. Kahn. On the uniqueness of matroid representations over GF(4). *Bull. London Math. Soc.*, 20:5–10, 1988.

[18] A. Kotzig. Quelques remarques sur les transformations κ. séminaire Paris, 1977.

[19] J. R. Stembridge. Nonintersecting paths, pfaffians, and plane partitions. *Advances in Math.*, 83:96–131, 1990.

[20] W. T. Tutte. A homotopy theorem for matroids, I, II. *Trans. Amer. Math. Soc.*, 88:144–174, 1958.

[21] W. T. Tutte. Lectures on matroids. *J. Res. Nat. Bur. Standards Sect. B*, 69b:1–47, 1965.