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Extraneous Fixed Points, Basin Boundaries and Chaotic Dynamics for Schröder and König Rational Iteration Functions

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Summary. The Schröder and König iteration schemes to find the zeros of a (polynomial) function $g(z)$ represent generalizations of Newton's method. In both schemes, iteration functions $f_m(z)$ are constructed so that sequences $z_{n+1}=f_m(z_n)$ converge locally to a root z^* of $g(z)$ as $O(|z_n-z^*|^m)$. It is well known that attractive cycles, other than the zeros z^*, may exist for Newton's method $(m=2)$. As m increases, the iteration functions add extraneous fixed points and cycles. Whether attractive or repulsive, they affect the Julia set basin boundaries. The König functions $K_m(z)$ appear to minimize such perturbations. In the case of two roots, e.g. $g(z)=z^2-1$, Cayley's classical result for the basins of attraction of Newton's method is extended for all $K_m(z)$. The existence of chaotic $\{z_n\}$ sequences is also demonstrated for these iteration methods.

Subject Classifications: AMS(MOS): 30D05, 30-04, 65E05, 65H05; CR: G.1.5.

1. Introduction

The Newton iteration function associated with a function $g(z)$,

$$N(z)=z-g(z)/g'(z), \tag{1.1}$$

defines a discrete dynamical system, $z_{n+1}=N(z_n)$, for which the sequences $\{z_n\}$ converge locally to a root z^* of $g(z)$ as $|z_{n+1}-z^*|=O(|z_n-z^*|^2)$. When $g(z)$ is a polynomial, as will be assumed here, $N(z)$ is a rational function. Cayley [5] posed the following question before the classical works of Julia [17] and Fatou [11]: What is the set of all initial values $z_0\in\mathbf{C}$ for which the sequence $\{z_n\}$ converges to a given root z^*; i.e. what is the *basin of attraction* of z^*? From the Julia-Fatou theory of iterates of rational functions, it follows that the basins of attraction of all roots share a common boundary, the set of all points $z\in\mathbf{C}\cup\{\infty\}$, in all of whose neighbourhoods the family of $N(z)$ and its iterates fails to be normal [1]. This set, now referred to as the *Julia set* $J(N)$ of $N(z)$, is a perfect set with generally nonintegral Hausdorff-Besicovitch dimension [10], i.e. a *fractal* [19]. Since $N(J)=J=N^{-1}(J)$, if the initial value $z_0 \in J$, then $z_n \in J$, i.e. the sequence never converges to a root z^*. In fact, it behaves chaotically.

Obviously, the possible nonconvergent behaviour of Newton sequences is worrisome when one considers iterative methods of solving nonlinear equations in general. It becomes important to ask: what is the set of points $z_0 \in \mathbf{C}$ for which the sequence $\{z_n\}$ does *not* converge to a root z^*? Clearly $J(N)$ is a subset. However, $J(N)$ is the closure of all repelling cycles of $N(z)$ so the z_n will never be attracted to it. In finite precision calculations, if $z_0 \in J$, then the z_n will eventually be thrown off J due to roundoff error. A greater threat is the existence of attractive cycles of $N(z)$ which may trap the sequence $\{z_n\}$. This was indeed recognized by Barna [2], who was primarily concerned with the behaviour of Newton sequences on the real line $\mathbf{R}$. One of his classical results states that if all roots of $g(z)$ are real, then all higher cycles on $\mathbf{R}$ are non-attractive. (He did, however, demonstrate attractive cycles in the complex plane.) Since his work, the dynamics of Newton's method on $\mathbf{R}$ has received much attention [16, 21]. Patterns of nonconvergent Newton sequences in the complex plane were shown by Curry [6], using a one-parameter family of cubic polynomials. Similar phenomena were observed [24] for the generalized family of Schröder iteration functions $S_m(z)$, constructed so that iteration sequences $z_{n+1}=S_m(z_n)$ converge locally to a root z^* of $g(z)$ as $O(|z_n-z^*|^m)$. For $m>2$, the $S_m(z)$ functions have extraneous fixed points, i.e. fixed points which are not roots of $g(z)$. Whether attractive or repulsive, their presence affects the global iteration dynamics. These aspects, also present for the König functions, will be discussed below. We also mention that Howland and Vaillancourt [14] studied the existence of attractive cycles for Newton functions which are meromorphic – a class of functions which have not yet received as much attention as rational and transcendental functions.

In Sect. 2, we outline the major ideas of iteration functions of prescribed order and introduce the Schröder and König methods, the latter of which will be the focus of this study. Aspects of Julia-Fatou theory relevant to iteration functions are presented in Sect. 3. In the specific case of two roots, Cayley's classical result [5] for Newton's method, i.e. that the basin boundary is the right bisector of the line joining the two roots, is shown to hold for all König functions $K_m(z)$. (This is *not* the case for the Schröder $S_m(z)$ functions.) We also present some results for the $K_m(z)$ applied to the functions $g_n(z)=z^n-1$. These results, along with other numerical evidence, indicate that basin boundary "interference" caused by extraneous fixed points of the $K_m(z)$ is minimized – these points are, in a sense, located as far as possible from the superattractive fixed points z_j^*. In Sect. 4 we investigate the König method as applied to the one-parameter family of cubic polynomials employed in [6] (Newton method) and [24] (Newton and Schröder methods). Regions in this "parameter space" are located which exhibit the morphology and dynamical patterns associated with the classical Mandelbrot sets of quadratic maps [8, 18]. As the parameter is varied continuously, the asymptotic behaviour of nonconvergent sequences

$\{z_n\}$ exhibits a cascade of period-doubling bifurcations eventually leading to chaos [7], a characteristic feature of quadratic and polynomial-like maps [9]. Detailed discussions on Newton's method as a chaotic dynamical system are found in [16] and [21].

2. Iteration Functions of Prescribed Order

Let $f(z)$: $\mathbf{C} \to \mathbf{C}$ be analytic on a compact subset $\mathcal{T}$ of the complex plane $\mathbf{C}$, having fixed point $p \in \mathcal{T}$, i.e. $f(p)=p$. The fixed point p is *attractive*, *indifferent* or *repulsive* depending on whether $|f'(p)|$ is less than, equal to or greater than one. If $f'(p)=0$, then p is *superattractive.* Given a starting value, or *seed* $z_0 \in \mathcal{T}$, we define the iteration sequence $\{z_n\}_0^\infty$ by $z_{n+1}=f(z_n)$, $n=0, 1, 2, \ldots$. Now assume that p is attractive and that $z_n \to p$ as $n \to \infty$. Let $e_n = z_n - p$ be the error associated with the nth iterate. Using the Taylor expansion of $f(z)$ about $z=p$, we have

$$\begin{aligned} e_{n+1} &= z_{n+1} - p \\ &= f(e_n + p) - f(p) \\ &= \frac{1}{(m!)} f^{(m)}(p)(e_n)^m + O[(e_n)^{m+1}], \qquad n \to \infty, \end{aligned} \tag{2.1}$$

where m is the smallest integer for which $f^{(m)}(p) \neq 0$. Then, $f(z)$ is said to be an *iteration function of order m.* A detailed study and classification of rational functions constructed with a specific number M of parameters to converge to a given number n of distinct complex points with a specified order σ was made by Smyth [23]. Newton's method corresponds to the special case $\sigma = 2$, with deg(numerator) = deg(denominator) + 1. The Schröder and König functions discussed below are subclasses of more general families of functions for $\sigma \geqq 2$.

The *Schröder iteration functions* [13, 22] are a generalization of Newton's method:

$$S_m(z) = z + \sum_{n=1}^{m-1} c_n[-g(z)]^n, \qquad m = 2, 3, 4, \ldots, \tag{2.2}$$

where the coefficients $c_n(z)$ are given by

$$c_n(z) = \frac{1}{n!}\left[\frac{1}{g'(z)}\frac{d}{dz}\right]^{n-1} \frac{1}{g'(z)}. \tag{2.3}$$

If $g(z)$ is assumed analytic in $\mathcal{T}$ and $g'(z) \neq 0$, then the $c_n(z)$, and hence the $S_m(z)$, are analytic in $\mathcal{T}$. The iteration sequences defined by $z_{n+1} = S_m(z_n)$ converge locally to a zero $z^* \in \mathcal{T}$ of $g(z)$ as $O(|z_n - z^*|^m)$, since $S_m(z^*) = z^*$ and

$$S'_m(z^*) = S''_m(z^*) = \ldots = S_m^{(m-1)}(z^*) = 0. \tag{2.4}$$

For a proof of Equation (2.4), see ([13], p. 520).

The $S_m(z)$ functions are truncations of an infinite series in $g(z)$, the first three terms of which are given below:

$$\mathbf{S}(z)=z-\frac{1}{g'(z)}\,g(z)-\frac{g''(z)}{2[g'(z)]^3}\,[g(z)]^2 \\ -\frac{\frac{1}{2}[g''(z)]^2-\frac{1}{6}g'(z)\,g'''(z)}{[g'(z)]^5}\,[g(z)]^3\ldots. \tag{2.5}$$

The construction of $S_m(z)$ requires a knowledge of the first $m-1$ derivatives of $g(z)$.

Only for $m=2$, Newton's method, does the fixed point condition $S_m(p)=p$ imply that $g(p)=0$. For $m>2$, it implies that either (i) $g(p)=0$, or (ii) $T_m(p)=0$, where

$$T_m(z)=\sum_{n=0}^{m-2} c_{n+1}(z)[-g(z)]^n. \tag{2.6}$$

We shall refer to zeros of the $T_m(z)$ which are not roots z_i^* of $g(z)$ as *extraneous fixed points*. Their appearance may complicate the root-finding procedure. As attractive fixed points, they may trap an iteration sequence, giving erroneous results for a root z^* of $g(z)$. Even as repulsive or indifferent fixed points, however, they may alter the structure of the *basins* of *attraction* for the roots. A seed z_0 which may be relatively close to one of the roots may, in fact, converge to another, remote root. In [24], these aspects of the Schröder functions were observed in an application to the family of functions $g_n(z)=z^n-1$.

The *König iteration functions* [15] corresponding to $g(z)$ are given by

$$K_m(z)=z+(m-1)\,\frac{[1/g(z)]^{(m-2)}}{[1/g(z)]^{(m-1)}}. \tag{2.7}$$

If $g(z^*)=0$, then $K_m^{(i)}(z^*)=0$ for $i=1, 2, \ldots, m-1$, as can be shown expanding $g(z)$ as a Taylor series about $z=z^*$. The case $m=2$ again corresponds to Newton's method. The functions $K_m(z)$ for $m=3$ and 4 are presented below:

$$K_3(z)=z+\frac{2gg'}{gg''-2(g')^2} \\ K_4(z)=z+\frac{3g[gg''-2(g')^2]}{6(g')^3-6gg'g''+g^2g'''}. \tag{2.8}$$

For $m>2$, as in the Schröder case, roots of $g(z)$ are necessarily fixed points of $K_m(z)$, *but not vice versa.* If we assume that all roots are simple zeros, then let

$$h(z)=[g(z)]^{-1}=\sum_{i=1}^{n} A_i(z-z_i^*)^{-1}, \tag{2.9}$$

so that

$$h^{(m)}(z)=(-1)^m\,m!\sum_{i=1}^{n} A_i(z-z_i^*)^{-(m+1)}.$$

Substitution into Equation (2.6) gives

$$K_m(z) = z - g(z)\, L_{m-1}(z)[L_m(z)]^{-1}, \tag{2.10}$$

where

$$L_m(z) = \sum_{i=1}^{n} A_i \prod_{j \neq i}^{n} (z - z_j^*)^m. \tag{2.11}$$

For $m > 2$, the fixed point condition $K_m(p) = p$ implies either (i) $g(p) = 0$, or (ii) $L_{m-1}(p) = 0$. Again the extraneous fixed points in (ii) may be either attractive or repulsive. In the case of the K_3 method, a stronger statement may be made:

Proposition 2.1. *All fixed points of $K_3(z)$ which are not roots of $g(z)$ are repulsive.*

Proof. A fixed point $K_3(p) = p$ implies either (i) $g(p) = 0$ or (ii) $g'(p) = 0$. Since $K_3'(z) = g^2[3(g'')^2 - 2g'g''']/[gg'' - 2(g')^2]^2$ (cf. Equation (2.8)), then condition (ii) implies that $K_3'(p) = 3$.

3. Julia-Fatou Theory and Rational Iteration Functions

The theory of iteration of rational functions originated with the classical research of Julia [17] and Fatou [11]. Details of important proofs are given by Brolin [4]. An excellent review including more recent work on complex analytic dynamics is given by Blanchard [3]. The book by Devaney [7] is also highly recommended. Here are outlined some important concepts relevant to rational root-finding functions.

Let $R(z)$ be a rational function, $R(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials with complex coefficients and no common factors and define its degree as $d = \deg(R) = \max\{\deg(P), \deg(Q)\} \geq 2$. Define the sequence of iterates $\{R^n(z)\}$ of $R(z)$ as

$$R^0(z) = z, \quad R^1(z) = R(z), \ldots, R^{n+1}(z) = R(R^n(z)), \quad n = 0, 1, 2, \ldots.$$

The inverses of $R(z)$ will be denoted by $R_i^{-1}(z)$, where the index $i = 1, 2, \ldots, d$ enumerates all branches. We consider $R(z)$ as a mapping on the compact Riemann sphere $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ with appropriate spherical metric. Given a point $z_0 \in \bar{\mathbf{C}}$, the *forward orbit* of z_0, $O^+(z_0)$, is given by the iteration sequence

$$z_{n+1} \uparrow R(z_n) = R^{n+1}(z_0).$$

If $R^k(p) = p$ and $R^m(p) \neq p$ for $m < k$, then p is a *fixed point of order k*. The set of distinct points $\{p_i, i = 1, 2, \ldots, k\}$, where

$$p_1 = R(p), \quad p_2 = R(p_1), \ldots, p_k = R(p_{k-1}),$$

is called a *k-cycle*. (If $k = 1$, then p is simply a fixed point of $R(z)$.) The k-cycle is *attractive*, *indifferent* or *repulsive*, depending on whether the multiplier $|[R^k(p)]'| = |R'(p_1)\, R'(p_2) \ldots R'(p_k)|$ (Chain Rule) is less than, equal to or greater than one, respectively.

Critical points $z = c_i$ of $R(z)$ are those points for which the equation $R(z) = v$ admits multiple roots. Locally, $R'(c_i) = 0$.

The *Julia set* $J(R)$ of $R(z)$ is classically defined as the set of $z\in\bar{\mathbf{C}}$ for which the family of iterates $R^n(z)$ is not normal in the sense of Montel [1]. By Arzela's theorem [1], $z\in J$ if the R^n are not equicontinuous in any neighbourhood of z. An equivalent and perhaps more working description is that $J(R)=$ closure {set of all repulsive k-cycles of $R(z)$, $k=1, 2, \ldots$}. Its complement is called the *Fatou set*, $F(R)=\bar{\mathbf{C}}\setminus J(R)$.

Important properties of $J(R)$ include: (i) $J\neq\phi$, (ii) $R(J)=J=R^{-1}(J)$, (iii) if J has interior points, then $J=\bar{\mathbf{C}}$, (iv) J is a perfect set [3, 4, 7]. In general $J(R)$ has non-integral Hausdorff-Besicovitch dimension D_{HB} [10].

Let p represent an attractive fixed point (or cycle) of $R(z)$. Its *basin of attraction* (stable set) $W(p)$ is defined as

$$W(p)=\{z\in\bar{\mathbf{C}}: R^n(z)\rightarrow p \text{ as } n\rightarrow\infty\}.$$

Illustrative Example: $R(z)=z^2$. The Julia set $J(R)$ is the unit circle $\mathscr{C}=\{z: |z|=1\}$. $R(z)$ has two superattractive fixed points, $p_1=0$ and $p_2=\infty$. All other fixed points of $R^n(z)$, $n=1, 2, \ldots$ form a dense set on $\mathscr{C}$ and are repulsive. $\mathscr{C}$ is the boundary of the two basins of attraction $W(0)=\{z: |z|<1\}$ and $W(\infty)=\{z: |z|>1\}$. Here $D_{HB}=1$. (In fact, $\mathscr{C}$ is the Julia set for $R(z)=z^n$, $n\geqq 2$.)

The following theorems play an important role in the behaviour of Newton-like sequences:

Theorem 3.1 (Fatou). *Every attractive cycle of $R(z)$ attracts at least one critical point.*

For an iteration function of order $m\geqq 2$, constructed to determine the roots of a polynomial $g(z)$, it follows from Equation (2.1) that the set of its critical points includes the roots z_i^* of $g(z)$.

Theorem 3.2 (Montel). *Given any neighbourhood U of a point $z\in J(R)$, the set of values $\bigcup_{n=1}^{\infty}\{R^n(U)\}$ omits at most two values $a, b\in\bar{\mathbf{C}}$.*

Montel's remarkable theorem implies that if $R(z)$ is a Newton-like rational function with attractive fixed points (or cycles) p_i, every neighbourhood of a point $z\in J(R)$ must contain open sets which belong to each basin of attraction $W(p_i)$. It then follows that all basins share a common boundary, $J(R)$. This accounts for the infinitely self-similar, fractal patterns observed in basin boundary plots for root-finding Newton methods. We first analyze the Newton, Schröder and König methods as applied to the relatively simple case of two roots.

The Case of Two Roots

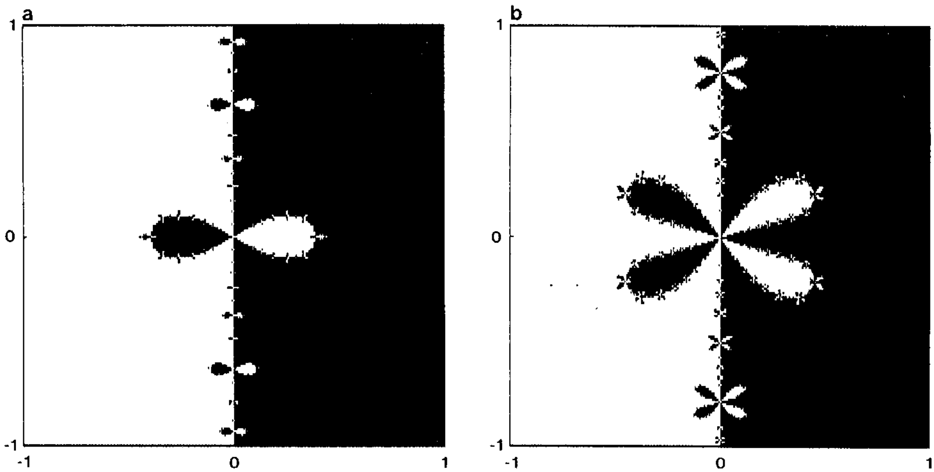
Without loss of generality, it suffices to consider $g(z)=z^2-1$. The associated Newton function, $N(z)=z/2+1/(2z)$, was analyzed by Cayley [5], who found that the boundary of the two attractive basins $W(1)$ and $W(-1)$ was the imaginary axis $\mathscr{I}$. Let $\phi(z)=(z+1)/(z-1)$ and $\mathscr{L}=\{z: \mathrm{Re}(z)<0\}$ and $\mathscr{R}=\{z: \mathrm{Re}(z)>0\}$. Then $\phi N\phi^{-1}(w)=w^2$, $\phi(\mathscr{L})=\{w: |w|<1\}$, $\phi(\mathscr{R})=\{w: |w|>1\}$, and $\phi(\mathscr{I})=\mathscr{C}$, the unit circle. In other words, Newton's function $N(z)$ is topologically conjugate to the map $R(w)=w^2$ on the w-sphere. Hence, $J(N)=\mathscr{I}$.

An extension of the above conjugacy analysis reveals the following property of König functions $K_m(z)$ for the case of two roots.

Proposition 3.1. *For* $g(z)=z^2-1$, $J(K_m)=\mathscr{I}$, $m=2, 3, 4, \ldots$, *where* $\mathscr{I}$ *denotes the imaginary axis.*

Proof. Since $h(z)=[g(z)]^{-1}=[(z-1)^{-1}-(z+1)^{-1}]/2$, and $h^{(m)}(z)$ given accordingly, it is straightforward to show that $K_m=(1+w^m)/(1-w^m)$, where $w=\phi(z)=(z+1)/(z-1)$. We find easily that $\bar{K}_m(w)=\phi K_m \phi^{-1}(w)=w^m$. Since $J(\bar{K}_m)=\mathscr{C}$ for $m \geqq 2$, the proposition follows.

It follows that all extraneous fixed points for the König functions are repulsive and lie on the imaginary axis $\mathscr{I}$. The Schröder functions do not behave in the same manner [24]. $S_3(z)$ has two repulsive fixed points at $z=\pm(1/5)^{1/2}$, and $S_4(z)$ has four repulsive fixed points at $z=\pm(2\pm i\sqrt{7})^{1/2}/\sqrt{11}$. These repulsive fixed points must lie on the respective Julia sets $J(S_m)$. As a result, the stable sets $W(\pm 1)$ are distortions of the Newton case, as the imaginary axis no longer serves as the common boundary. This is seen in Fig. 1, where basin



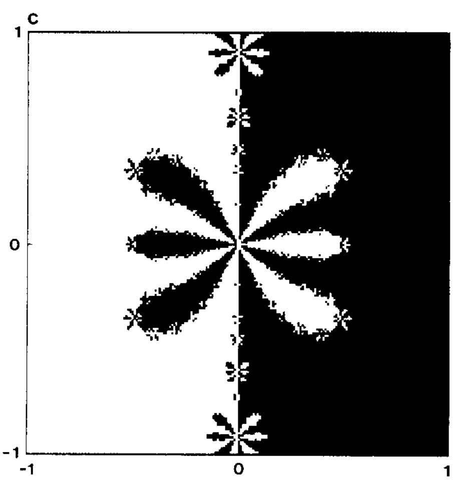


Fig. 1. Basins of attraction for the Schröder $S_m(z)$ functions corresponding to $g(z)=z^2-1$ in the complex region $[-1, 1]\times[-1, 1]$; $W(-1)$ in white, $W(1)$ in black.
(a) S_3, **(b)** S_4, **(c)** S_5

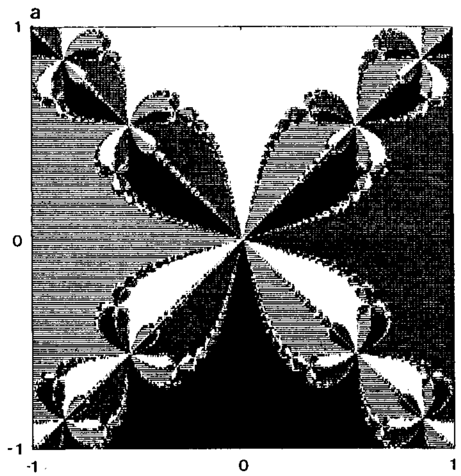
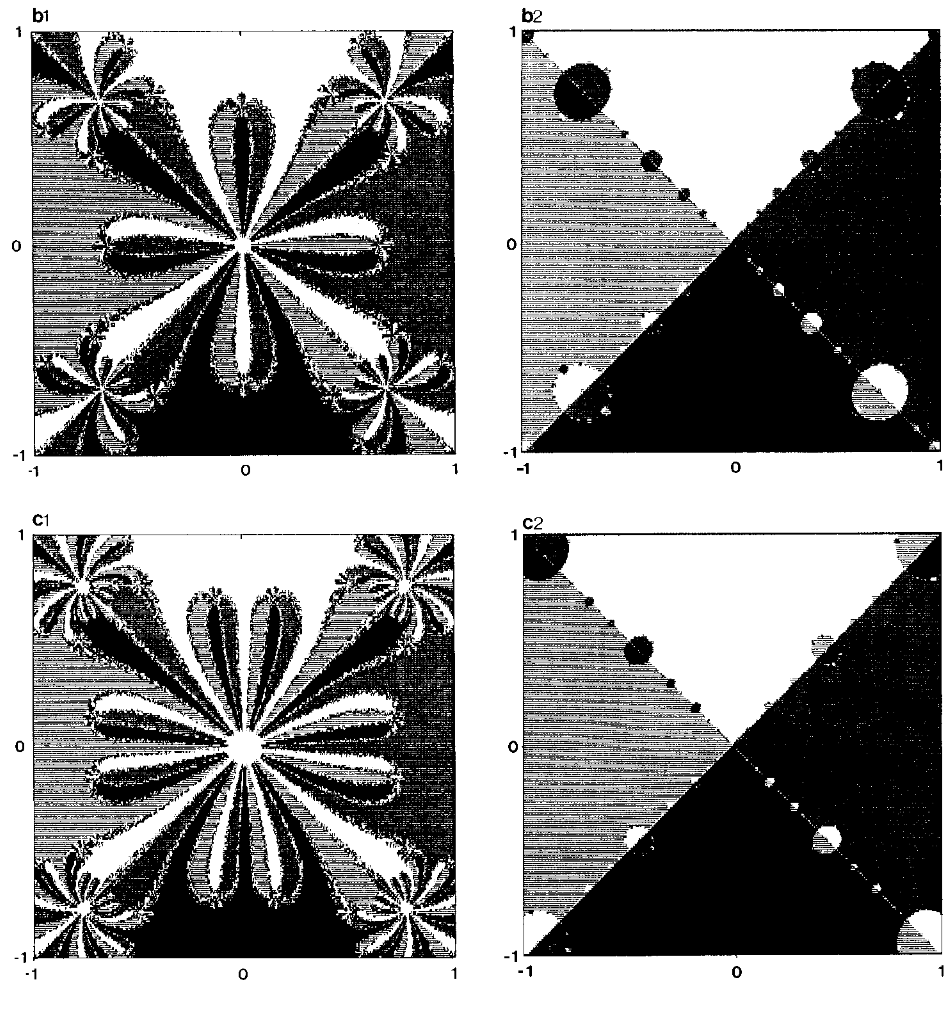


Fig. 2. A comparison of basins of attraction for Schröder and König functions applied to $g(z) = z^4 - 1$ in the complex region $[-1, 1] \times [-1, 1]$; $W(i)$ white; $W(-i)$ black; $W(-1)$ light grey; $W(1)$ dark grey.
(a) $S_2 = K_2 =$ Newton;
(b1) S_3; **(b2)** K_3;
(c1) S_4; **(c2)** K_4



plots are presented for $S_m(z)$, $m=3, 4, 5$. For example, in the case $m=3$, a large interfering component of $W(-1)$ is seen to extend from the origin toward the root $z^*=1$, and *vice versa*. It is natural to conjecture that for general m, $S_m(z)$ introduces $m-2$ repulsive fixed points into each of the right and left half-planes, symmetrically about $z=0$. The asymptotic distribution of these points as $m\to\infty$ is an interesting question.

More Than Two Roots

Our attention is restricted to the family of functions $g_n(z)=z^n-1$. Some definite results for these cases illustrate the minimal interference caused by the Julia set boundaries $J(K_m)$. It is expected that similar effects exist in the case of less symmetric distributions of roots.

Proposition 3.2. *For* $n\geqq 2$, *the set of roots* z_j^* *of* $g_n(z)=z^n-1$ *is* $Z_n=\{z_j^*=\exp(2j\pi/n),\ j=0, 1, 2, \dots, n-1\}$. *For* $m\geqq 3$, *all extraneous fixed points* p *of* $K_m(z)$, *i.e.* $p\notin Z_n$, *are repulsive and lie on the rays* $\arg(z)=(2j+1)\pi/n$.

Proof. Given in Appendix.

Geometrically, for a given value of $n\geqq 2$, the repulsive fixed points of $K_m(z)$ lie on the perpendicular bisectors of the regular n-gon formed by the nth roots of unity. This result is again in sharp contrast with the Schröder method applied to the $g_n(z)$ [24]. For $m=3$ and 4, the $S_m(z)$ introduce $m-2$ repulsive fixed points into each sector $(2j-1)\pi/n<\arg(z)<(2j+1)\pi/n$, $j=0, 1, \dots, n-1$. Again, we conjecture that this procedure continues for $m>4$. Attractive basin plots for the $S_m(z)$ and $K_m(z)$ associated with $g_4(z)$, $m=3$ and 4, are presented in Fig. 2, again demonstrating the relative minimization of interference afforded by the König procedure. It would be interesting to compare numerical estimates of the fractal dimensions of the Julia set basin boundaries for both methods.

4. Parameter Space and Chaotic Dynamics

As mentioned earlier, root-finding methods involving rational functions are not guaranteed to converge to the zeros of a function $g(z)$. If the initial point $z_0\in J$, the Julia set, then the sequence $\{z_n\}$ will always remain on J. (In practice, round-off error will eventually kick the sequence away from the repeller set J.) Apart from this rather improbable case, the possibility does exist that the z_n converge to periodic cycles or even exhibit chaotic behaviour. Instead of constructing specific examples of such pathological behaviours, we may systematically examine iteration schemes associated with a parameterized family of polynomials. Here, we consider the one-parameter family of cubic polynomials

$$g_A(z)=z^3+(A-1)z-A, \tag{4.1}$$

the roots of which are $z_1^*=1$, $z_{2,3}^*=(-1\pm\sqrt{1-4A})/2$. We shall now work in the complex-parameter space $A\in\mathbf{C}$. For a given rational iteration function, each point $A=(\mathrm{Re}(A), \mathrm{Im}(A))$ represents a dynamical system with its own cycles and Julia set. It will be seen that pathological behaviour will be concentrated in particular regions of parameter space whose morphology and dynamics are similar to those of the classical Mandelbrot sets of complex quadratic mappings [19, 9].

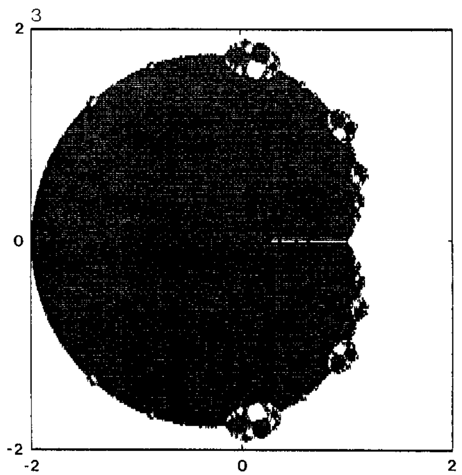
Curry [6] first examined Newton's method in this parameter space to discover regions where extraneous attractive periodic cycles exist. This feature is also observed for the Schröder functions associated with $g_A(z)$ [24]. In this latter case, there may exist attractive fixed points corresponding to the zeroes of $T_m(z)$ in equation (2.6). Here we include some aspects of Newton's method in parameter space and present results for S_3 and K_3 methods.

To detect the existence of extraneous attractive cycles for $S_m(z)$ and $K_m(z)$, we observe the orbits of their critical points. By Fatou's Theorem 3.1, each attractive cycle will attract at least one critical point. Obviously, only critical points which are not roots of $g(z)$, which we shall refer to as *free critical points*, could possibly detect these extra cycles.

In the parameter space plots presented below, regions in the complex A-plane are shaded according to where the relevant free critical point is attracted: white for $z_1^*=1$, grey for $z_{2,3}^*$ and black for neither. Details of the microcomputing involved in generating these plots are given in [24].

Newton's Method

The only free critical point is $c=0$. The parameter space plot for the Newton method in Fig. 3 was first presented in [6]. Small black areas, representing A-values for which pathological attractive cycles exist, are observed at $A \simeq (0.31,$



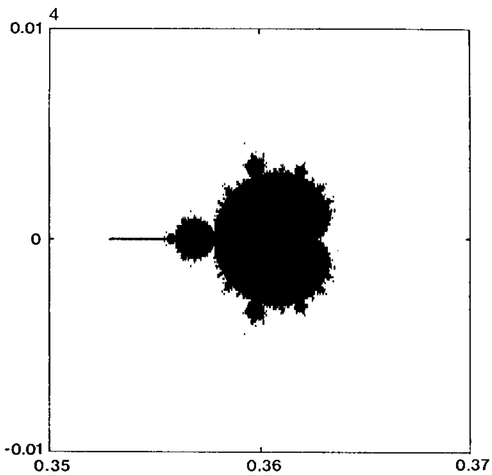


Fig. 3. Complex A-parameter space plot for Newton's method applied to the polynomials $g_A(z)$ in Equation (4.1): regions in $A \in [-2, 2] \times [-2, 2]$ for which the free critical point $c=0$ converges to $z_1^*=1$ (white), z_2^* or z_3^* (grey) or none of these roots (black)

Fig. 4. A magnification of the Mandelbrot-like set in the region $[0.35, 0.37] \times [-0.01, 0.01]$ of Fig. 3, i.e. parameter values A for which the critical point $c=0$ does not converge to a root z_i^* of $g_A(z)$

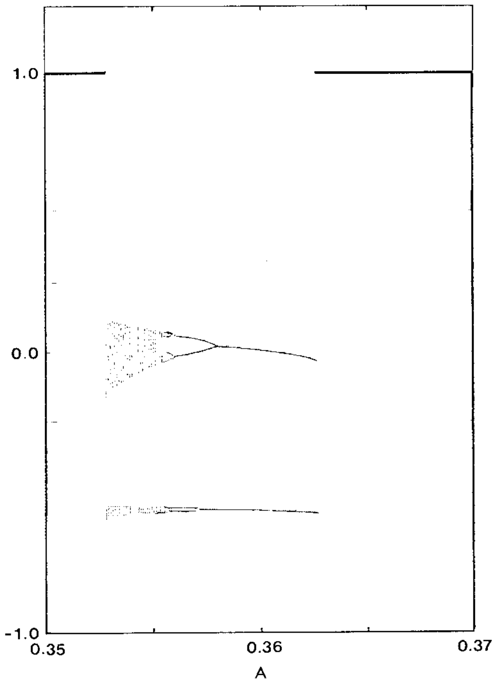
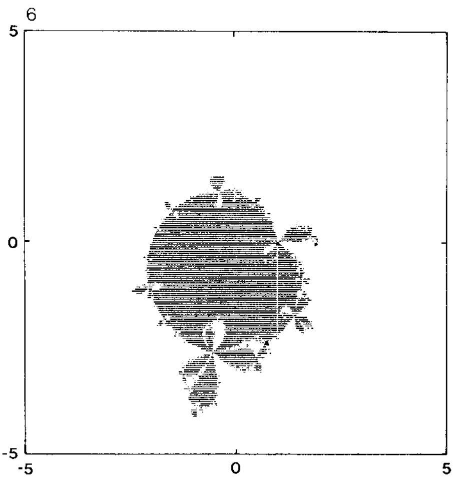


Fig. 5. Asymptotic trajectory of the critical point $c=0$ for Newton's method applied to $g_A(z)$ for $0.35 \leqq A \leqq 0.37$, i.e. the real A-values of Fig. 4

± 1.64) and (1.01, ± 0.98). When magnified, these regions have the same general shape as the remarkable Mandelbrot sets [18, 20] for quadratic maps $R(z)=z^2+c$. Four other sets symmetric about the real A-axis are detectable at the real values $A \simeq 0.26$, 0.36, 0.5 and 0.65. Note that real attractive cycles lying in these regions do not conflict with Barna's theorem for Newton's method on the real line: in all cases, the two roots $z^*_{2,3}$ are complex. Figure 4 gives a magnification of the region $[0.35, 0.37] \times [-0.01, 0.01]$. As in the case of quadratic maps, these sets represent zones of stable cycles which undergo the classical period-doubling route to chaotic behaviour [12]. For example, the major cardioid in Fig. 4 represents A-values for which there exist attractive 2-cycles. (Recall that no extraneous fixed points can occur for Newton's method.) The adjacent circular region corresponds to attractive 4-cycles, etc. In Fig. 5, we plot the asymptotic orbits, (z_n for $n > 10000$) of the critical point $c_1 = 0$ for the range of real parameter values $0.35 \leqq A \leqq 0.37$. As the parameter A is decreased from 0.37, a transition is observed at $A = 0.362683\ldots$, when c_1 becomes mapped to attractive 2-cycles. The cascade to 2-cycles proceeds quite rapidly, with an eventual transition to chaotic behaviour. A Sarkovskii-type ordering of cycles [7] with the appearance of a 3-cycle is observed. At $A = 0.35286\ldots$, a return from chaotic behaviour back to the fixed point $z^*_1 = 1$ is observed.

S_3 *Iteration*

Figure 6 is the parameter space plot obtained with the free critical point $c_1 = [(A-1)/15]^{1/2}$. The parameter space map for the other critical point $c_2 = -c_1$



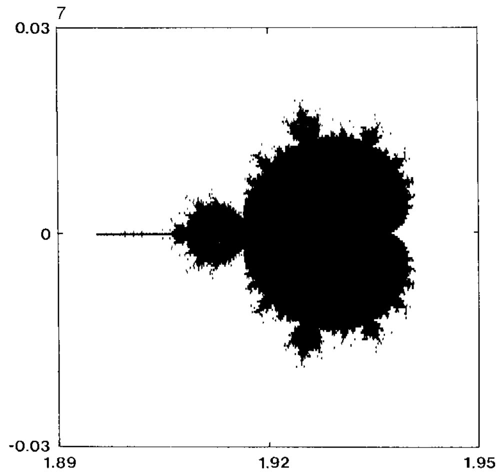


Fig. 6. Complex A-parameter space plot for the Schröder S_3 method applied to the polynomials $g_A(z)$: regions in $[-5, 5] \times [-5, 5]$ for which the free critical point $c_1 = \sqrt{(A-1)/15}$ converges to $z_1^* = 1$ (white), z_2^* or z_3^* (grey), or none of these roots (black)

Fig. 7. A magnification of a Mandelbrot-like set in Fig. 6 for the S_3 method

is obtained by reflecting the regions in Fig. 6 about the real A-axis. An enlargement of the Mandelbrot-like set in $[1.89, 1.95] \times [0, 0.03]$ is shown in Fig. 7 along with its reflection about the real A-axis. The upper half of this set, including the real axis comes from the c_1 plot. The major cardioid corresponds to attractive fixed points, or zeros of $T_3(z)$ in Equation (2.6) for $g(z) = g_A(z)$. A little algebra shows that such attractive fixed points can exist for parameter values A where a root of the equation

$$T_3(z) = 12z^4 + 9(A-1)z^2 - 3Az + (A-1)^2 = 0$$

satisfies the inequality

$$|S_3'(z)| = |5 + (1-A)/(3z^2)| < 1.$$

Figure 8 shows the asymptotic orbits of c_1 in the real parameter region $1.89 \leqq A \leqq 1.95$.

K_3 *Iteration*

In Fig. 9 the region of complex A-space $[-3, 3] \times [-4, 2]$ has been probed with the free critical point $c_1 = [(A-1)/15]^{1/2}$. (The other free critical point is $c_2 = -c_1$. Its corresponding parameter space plot is again a reflection of Fig. 9 about the real-A axis.) Only two regions of nonconvergence to the roots of $g_A(z)$ are detectable at this resolution: at $A \simeq (1.99, -3.26)$ and $(-1.98, -2.68)$. A magnification of the former region is presented in Fig. 10. The primary car-

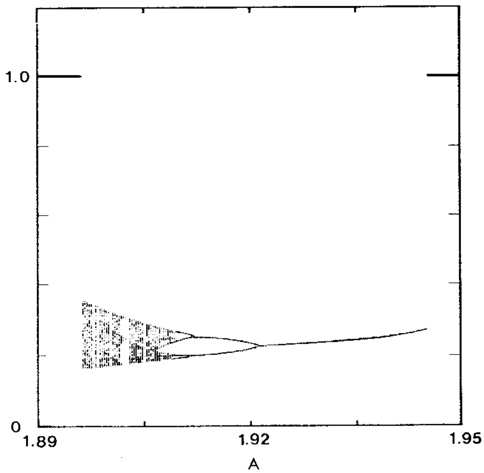
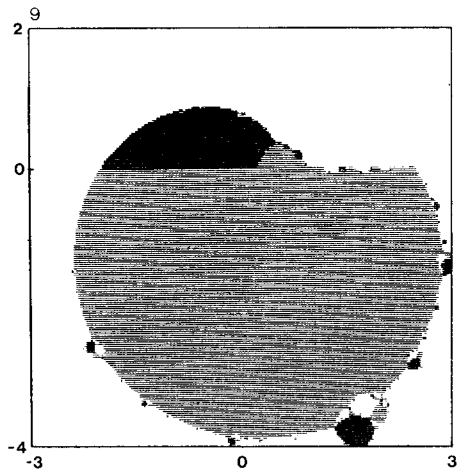


Fig. 8. Asymptotic trajectories of the S_3 critical point $c_1 = \sqrt{(A-1)/15}$ for the real A-values in Fig. 7, $1.89 \leqq A \leqq 1.95$



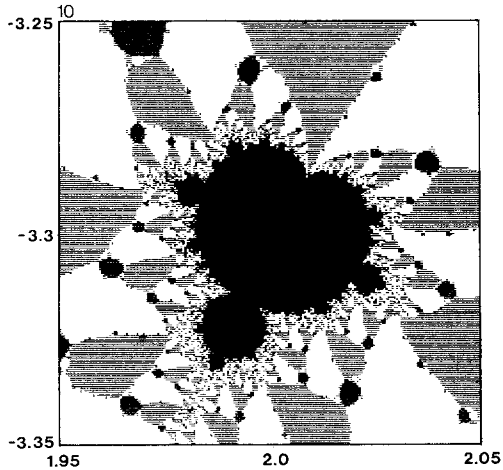


Fig. 9. Complex A-parameter space plot for the König K_3 method applied to the polynomials $g_A(z)$: regions in $[-3, 3] \times [-4, 2]$ for which the free critical point $c_1 = V(A-1)/15$ converges to $z_1^* = 1$ (white), z_2^* or z_3^* (grey) or none of the roots (black)

Fig. 10. A magnification of the region $[1.95, 2.05] \times [-3.35, -3.25]$ revealing a Mandelbrot-like set as well as the complicated dynamics occurring near this set

dioid in the Figure represents parameter values with attractive 2-cycles (cf. Proposition 2.1). The cascade of period-doubling bifurcations leading to chaotic behaviour has also been numerically observed here.

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Appendix

Proof of Proposition 3.2.

We introduce polynomials $p_k(x)$ defined by

$$\left[\frac{1}{z^n-1}\right]^{(k)}=\frac{p_k(z^n)}{z^k(z^n-1)^{k+1}} \tag{A.1}$$

so that $p_0(x)=1$.

The König functions $K_m(z)$ constructed from the polynomial $g(z)=z^n-1$ may now be written as

$$K_m(z)=z+(m-1)\,z(z^n-1)\,\frac{p_{m-2}(z^n)}{p_{m-1}(z^n)}. \tag{A.2}$$

Differentiation of Equation (A.1) gives the following recurrence relation for the $p_k(x)$,

$$p_{k+1}(x)=nx(x-1)\,p_k'(x)+[k-(nk+n+k)x]\,p_k(x). \tag{A.3}$$

The next three polynomials are given by

$$p_1(x)=-nx$$
$$p_2(x)=nx[(n+1)\,x+(n-1)]$$
$$p_3(x)=-nx[(n+1)(n+2)\,x^2+4(n+1)(n-1)\,x+(n-1)(n-2)].$$

Note that for all $n\geqq 2$, all roots x are real and nonpositive.

Proposition. *$p_k(x)$ is a polynomial of degree k with $p_k(x)\to\infty$ as $x\to-\infty$, and $p_k(x)$ has q zero roots and $k-q$ distinct negative roots where $q\geqq k/n$.*

Proof. By induction on k. For $k=1$, $p_1(x)$ has one zero root. Suppose inductively that $r_1<r_2<\ldots<r_{k-q}$ are the negative roots of $p_k(x)$. Then

$$p_k(x)=(-1)^k[Ax^k+\ldots+Bx^q], \qquad \text{where } A, B>0.$$

(By Descartes' Rule of Signs, all coefficients of $p_k(x)$ must have the same sign.) Using (A.3),

$$\begin{aligned} p_{k+1}(x)&=(-1)^k[(nk-(nk+n+k))\,Ax^{k+1}+\ldots+(-nq+k)\,Bx^q] \\ &=(-1)^{k+1}[(n+k)\,Ax^{k+1}+\ldots+(nq-k)\,Bx^q]. \end{aligned}$$

Therefore $p_{k+1}(x)$ is a polynomial of degree $k+1$, $p_{k+1}(x)\to\infty$ as $x\to-\infty$ and $p_{k+1}(x)$ has at least q zero roots. For any root r_i of $p_k(x)$ we have, by (A.3),

$$p_{k+1}(r_i)=n(n^2-r_i)\,p_k'(r_i),$$

which has the same sign as $p_k'(r_i)$. The signs of $p_k'(r_i)$, $i=1, 2, \ldots, k-q$ alternate as $-+-+\ldots$, so $p_{k+1}(x)$ has at least $k-q$ distinct negative roots which are less than r_{k-q}.

There is now only one root of the polynomial $p_{k+1}(x)$ unaccounted for. Hence, it must be a real root. Suppose that $nq-k>0$ so that $p_{k+1}(x)$ has exactly q zero roots. Now $p_{k+1}(r_{k-q})$ has a sign of $(-1)^{k-q}$ and $p_{k+1}(x)$ is approximately $(-1)^{k+1}(nq-k)Bx^q$ near $x=0$. Hence the remaining root must lie between r_{k-q} and 0. Thus there are exactly q zero roots and since $nq-k>0$ and the numbers are integers, $nq-k\geqq 1$; i.e. $q\geqq(k+1)/n$. If $nq-k=0$, then $p_{k+1}(x)$ has $q+1$ zero roots. The number of zero roots then satisfies $q+1>(k+1)/n$. The result now follows by induction.

Theorem. *The roots of $p_k(z^n)$ all have arguments of the form $(2j+1)\pi/n$.*

Proof. All roots of $p_k(x)$ have argument π, by the previous proposition.

By Equation (A.2), the above result establishes the location of the fixed points of the $K_m(z)$ which are not the nth roots of unity. (It also establishes the location of all poles.) To show that these fixed points are repulsive, differentiate Equation (A.2) with respect to z and set $z=r$, where $p_{m-2}(r^n)=0$ to give

$$K'_m(r)=1+(m-1)\,n r^n(r^n-1)\,p'_{m-2}(r^n)/p_{m-1}(r^n).$$

From the recurrence relation (A.3), we find that $K'_m(r)=m$.

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