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SUSPENSION OF THE LUSTERNIK-SCHNIRELMANN CATEGORY

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Let cat be the Lusternik-Schnirelmann category structure as defined by Whitehead [6] and let $\overline{\text{cat}}$ be the category structure as defined by Ganea [2]. We prove that

$$\Sigma\,\text{cat}\,X = \text{w}\,\Sigma\,\text{cat}\,X \text{ for any space } X$$

and

$$\Sigma\,\overline{\text{cat}}\,X = \text{w}\,\overline{\text{cat}}\,X \text{ for any simply connected } X.$$

It is known that $\text{w}\,\Sigma\,\text{cat}\,X = \text{conil}\,X$ for connected X. Dually, if X is simply connected,

$$\Omega\,\overline{\text{cocat}}\,X = \text{w}\,\overline{\text{cocat}}\,X.$$

1. We work in the category $\mathscr{T}$ of based topological spaces with the based homotopy type of CW-complexes and based homotopy classes of maps. We do not distinguish between a map and its homotopy class. Constant maps are denoted by 0 and identity maps by 1.

We recall some notions from Peterson's theory of structures [5; 1] which unify the definitions of the numerical homotopy invariants akin to the Lusternik-Schnirelmann category. For any category $\mathscr{C}$, by a right structure $\mathscr{R} = (R, P, T; d, j)$ over $\mathscr{C}$ we mean a triple R, P, T of covariant functors from $\mathscr{C}$ to $\mathscr{T}$ together with a pair of natural transformations $d\colon R \to P$ and $j\colon T \to P$. An object $X \in \mathscr{C}$ is said to be $\mathscr{R}$-structured if there exists a map $\phi\colon RX \to TX$ such that $jX \circ \phi \simeq dX$. If $\mathscr{R} = (R, P, T; d, j)$ is a right structure over $\mathscr{C}$, its suspension Σ is the right structure $(\Sigma R, \Sigma P, \Sigma T; \Sigma d, \Sigma j)$ over $\mathscr{C}$. The associated weak structure to $\mathscr{R}$ is the right structure $\text{w}\mathscr{R} = (R, P, T_{\text{w}}; d, j_{\text{w}})$ over $\mathscr{C}$ where we define $q\colon P \to Q$ to be the cofibre of j and $j_{\text{w}}\colon T_{\text{w}} \to P$ to be the fibre of q. Then $x \in \mathscr{C}$ can be $\text{w}\mathscr{R}$-structured if and only if $qX \circ dX \simeq 0$.

Let

$$N \to T' \xrightarrow{j'} P$$

be the natural fibration obtained from j and let

$$N \to M \xrightarrow{p} R$$

be the fibration obtained from pulling back j' by means of $d\colon R \to P$. Then we call $\overline{\mathscr{R}} = (R, R, M; 1, p)$ the strong structure associated with $\mathscr{R}$.

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Let $\mathscr{K}_n = (1, \Pi_1^n, T_1^n; \Delta, j)$ be the n-category structure over $\mathscr{T}$, where T_1^n is the fat wedge functor, Δ the diagonal transformation, and j the inclusion transformation. The X is $\mathscr{K}_n$-structured if and only if $\operatorname{cat} X < n$ using Whitehead's definition. Let $\overline{\operatorname{cat}}$ be the strong structure associated with cat. Then $\overline{\operatorname{cat}}$ is equivalent to Ganea's definition of category and $\operatorname{cat} X = \overline{\operatorname{cat}}\, X$ for $X \in \mathscr{T}$.

2. Let wΣ cat be the weak structure associated with Σ cat. (There is some confusion here in the literature. This invariant wΣ cat is denoted by Σ w cat in [**1**; **5**].)

THEOREM 2.1. *For any connected* $X \in \mathscr{T}$,

$$\Sigma \operatorname{cat} X = \mathrm{w}\, \Sigma \operatorname{cat} X = \operatorname{conil} X.$$

Proof. From the definitions, $\Sigma \operatorname{cat} X < n$ if and only if there exists a map $\phi\colon \Sigma X \to \Sigma T_1^n X$ such that $\Sigma j \circ \phi \simeq \Sigma \Delta$ and $\mathrm{w}\ \Sigma \operatorname{cat} X < n$ if and only if $\Sigma (q \circ \Delta) \simeq 0$, where $q\colon X^n \to X^{(n)}$ is the projection from the n-fold product to the n-fold smash product of X.

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\ \phi\ } & \Sigma T_1^n X \\ & \searrow^{\Sigma\Delta} & \downarrow{\scriptstyle \Sigma j} \\ & & \Sigma X^n \\ & & \downarrow{\scriptstyle \Sigma q} \\ & & \Sigma X^{(n)} \end{array}$$

Since $X^{(n)}$ is the cofibre of j, it follows that $\Sigma \operatorname{cat} X \geqq \mathrm{w}\ \Sigma \operatorname{cat} X$.

Suppose that $\mathrm{w}\ \Sigma \operatorname{cat} X < n$. Then there exist well-known maps

$$\chi\colon \Sigma X^n \to \Sigma T_1^n X \quad \text{and} \quad \tau\colon \Sigma X^{(n)} \to \Sigma X^n$$

such that $\chi \circ \Sigma j \simeq 1$, $\Sigma q \circ \tau \simeq 1$ and $\Sigma j \circ \chi + \tau \circ \Sigma q \simeq 1$. Let $\phi = \chi \circ \Sigma \Delta$ so that

$$\begin{aligned} \Sigma j \circ \phi &= \Sigma j \circ \chi \circ \Sigma \Delta \\ &\simeq \Sigma j \circ \chi \circ \Sigma \Delta + \tau \circ \Sigma q \circ \Sigma \Delta \\ &\qquad\qquad \text{since } \Sigma q \circ \Sigma \Delta \simeq 0 \\ &= (\Sigma j \circ \chi + \tau \circ \Sigma q) \circ \Sigma \Delta \\ &\qquad\qquad \text{since } \Sigma \Delta \text{ is a suspension} \\ &\simeq \Sigma \Delta. \end{aligned}$$

Hence $\Sigma \operatorname{cat} X < n$ and so $\Sigma \operatorname{cat} X = \mathrm{w}\ \Sigma \operatorname{cat} X$. The equality $\mathrm{w}\ \Sigma \operatorname{cat} X = \operatorname{conil} X$ for connected X follows from [**3**, Theorem 4.1].

THEOREM 2.2. *For any simply connected* $X \in \mathscr{T}$,

$$\Sigma \overline{\text{cat}}\, X = \text{w}\, \overline{\text{cat}}\, X.$$

Proof. Let the fibration

$$F_n \xrightarrow{i} E_n \xrightarrow{p} X$$

be the Whitney sum of n copies of the standard fibration $\Omega X \to PX \to X$ where PX is the space of paths in X starting at the base point. Let ϵ: $X \to C_n$ be the cofibre of p. Now $\overline{\text{cat}}\, X < n$ if and only if there exists a map r: $X \to E_n$ such that $p \circ r \simeq 1$. Hence, it follows that $\Sigma \overline{\text{cat}}\, X < n$ if and only if there exists a map s: $\Sigma X \to \Sigma E_n$ such that $\Sigma p \circ s \simeq 1$ and $\text{w}\, \overline{\text{cat}}\, X < n$ if and only if $\epsilon \simeq 0$.

Suppose that $\text{w}\, \overline{\text{cat}}\, X < n$ so that in the Barratt-Puppe sequence

$$E_n \xrightarrow{p} X \xrightarrow{\epsilon} C_n \xrightarrow{k} \Sigma E_n \xrightarrow{\Sigma p} \Sigma X \xrightarrow{\Sigma \epsilon} \Sigma C_n,$$

$\Sigma E_n \simeq C_n \vee \Sigma X$ and $\Sigma X \simeq \Sigma E_n \cup CC_n$. Hence, it is possible to find a map s: $\Sigma X \to \Sigma E_n$ such that $\Sigma p \circ s \simeq 1$ and so $\text{w}\, \overline{\text{cat}}\, X \geqq \Sigma \overline{\text{cat}}\, x$.

Conversely, suppose that $\Sigma \overline{\text{cat}}\, X < n$ so that there exists a map s: $\Sigma X \to \Sigma E_n$ such that $\Sigma p \circ s \simeq 1$. The map $\langle k, s \rangle$: $C_n \vee \Sigma X \to \Sigma E_n$ in which C_n is mapped by k and ΣX is mapped by s induces isomorphisms in homology. Since ΣE_n and C_n are simply connected, it follows from Whitehead's theorem that $\langle k, s \rangle$ is a homotopy equivalence. Hence $\epsilon \simeq 0$ and $\text{w}\, \overline{\text{cat}}\, X < n$ which proves the theorem.

If $\overline{\text{cocat}}$ is the structure defined by Ganea [**2**; § 6], Theorem 2.2 dualizes to give the following theorem.

THEOREM 2.3. *For any simply connected* $X \in \mathscr{T}$,

$$\Omega\, \overline{\text{cocat}}\, X = \text{w}\, \overline{\text{cocat}}\, X.$$

Remark 2.4. In the proof of Theorem 2.2, the only fact that we used about the $\overline{\text{cat}}$ structure was that d was the identity functor. Hence, if $\mathscr{R} = (R, R, T; 1, j)$ is a right structure over $\mathscr{C}$, for any $X \in \mathscr{T}$ such that TX and RX are simply connected, it follows that

$$X \text{ is } \Sigma\mathscr{R}\text{-structured if and only if } X \text{ is w}\,\mathscr{R}\text{-structured.}$$

Remark 2.5. Theorem 2.2 together with the results of [**4**] show that even though $\text{cat}\, X = \overline{\text{cat}}\, X$, it does not follow that $\Sigma\, \text{cat}\, X = \Sigma\, \overline{\text{cat}}\, X$ or that $\text{w cat}\, X = \text{w}\, \overline{\text{cat}}\, X$.

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