SOME PROBLEMS IN ALGEBRAIC TOPOLOGY ON LUSTERNIK-SCHNIRELMANN CATEGORIES AND COCATEGORIES

by

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ABSTRACT

In this thesis we are concerned with certain numerical invariants of homotopy type akin to the Lusternik-Schnirelmann category and cocategory.

In a series of papers I. Berstein, T. Ganea, and P. J. Hilton developed the concepts of the category and weak category of a topological space. They also considered the related concepts of conilpotency and cup product length of a space and the weak category of a map. Later T. Ganea gave another definition of category and weak category (which we will write as G-cat and G-wcat) in terms of fibrations and cofibrations and hence this dualizes easily in the sense of Eckmann-Hilton.

We find the relationships between these invariants and then find various examples of spaces which show that the invariants are all different except cat and G-cat. The results are contained in the following theorem. The map $e: B \longrightarrow \Omega\Sigma B$ is the natural embedding. All the invariants are normalized so as to take the value 0 on contractible spaces.

<u>Theorem.</u> Let B have the homotopy type of a simply connected CW-complex, then

cat B=G-cat B $\geqslant$ G-wcat B $\geqslant$ wcat B $\geqslant$ wcat e $\geqslant$ conil B $\geqslant$ $\cup$-long B

and furthermore all the inequalities can occur.

All the examples are spaces of the form $B = S^q \cup_\alpha e^n$ where $\alpha \in \pi_{n-1}(S^q)$. When B is of this form, we obtain conditions for the category and and weak categories of B to be less than or equal to one in terms of Hopf invariants of α. We use these conditions to prove the examples.

We then prove the dual theorem concerning the relationships between the invariants cocategory, weak cocategory, nilpotency and Whitehead product length.

<u>Theorem.</u> Let A be a countable CW-complex, then

cocat A $\geqslant$ wcocat A $\geqslant$ nil A $\geqslant$ W-long A

and furthermore all the inequalities can occur.

The proof is not dual to the first theorem, though the examples we use to show that the inequalities can exist are all spaces with two non-zero homotopy groups.

The most interesting of these examples is the space A with 2 non-zero homotopy groups, $\mathbb{Z}$ in dimension 2 and $\mathbb{Z}_4$ in dimension 7 and with k-invariant $u^4 \in H^8(\mathbb{Z},2;\mathbb{Z}_4)$. This space is not an H-space, but has weak cocategory 1. The condition wcocat $A \leqslant 1$ is equivalent to the fact that $d \simeq 0$ in the fibration $D \xrightarrow{d} A \xrightarrow{e} \Omega\Sigma A$. In order to show that wcocat $A = 1$ we have to calculate the cohomology ring of $\Omega\Sigma K(\mathbb{Z},2)$. The method we use to do this is the same as that used to calculate the cohomology ring of ΩS^{n+1} using James' reduced product construction.

Finally we show that for the above space A the fibration

$$\Omega A \xrightarrow{g} A^S \xrightarrow{f} A$$

has a retraction ρ such that $\rho \circ g \simeq 1$ even though A is not an H-space.