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NEWTON'S METHOD FOR MULTIPLE ROOTS

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Abstract—We investigate the basins of attraction in the complex plane of Newton's method for finding multiple roots and illustrate what happens as two simple roots coalesce to form a double root.

1. INTRODUCTION

Newton's method for finding a real or complex root of a function is very efficient near a simple root because the algorithm converges quadratically in the neighborhood of such a root. However, at a multiple root, that is, a root of order greater than one, Newton's method only converges linearly. Various modifications of Newton's method have been proposed that converge quadratically at multiple roots. One standard method that we discuss here will find the roots of the function $g(z)$ by applying Newton's method to the function $g(z)/g'(z)$. Newton's method applied to this quotient will always converge quadratically near all the roots of $g(z)$. This method introduces extraneous fixed points into Newton's method, but they are always repelling. We plot the basins of attraction for the roots in the complex plane and illustrate what happens to these basins as two simple roots coalesce to form a multiple root. We then compare this method with other Newton methods.

2. NEWTON'S METHOD FOR MULTIPLE ROOTS

The *standard Newton's Method* for finding the real or complex roots of a function $g(z)$ consists of iterating the function

$$N(z) = z - \frac{g(z)}{g'(z)}$$

by starting with some initial approximation z_0 and defining the $(n+1)$st approximation by $z_{n+1} = N(z_n)$. See [1] and [2] for the connection between the global study of Newton's method for finding complex roots and dynamics in the complex plane.

Any root of the function $g(z)$ is a fixed point of $N(z)$. If w is a root of a polynomial $g(z)$ of order k, then the *multiplier* (or *eigenvalue*) of the fixed point w is $N'(w) = (k-1)/k$; this multiplier gives information about the behavior of the dynamics of the iteration near the fixed point [3, §6.1]. At a simple root, $k = 1$ and the multiplier is zero, which means that w is a super-attracting fixed point and Newton's method converges quadratically in some neighborhood of the fixed point. At a multiple root, $k > 1$ and the multiplier lies between 0 and 1, which means that the fixed point w is attracting, but that the convergence will only be linear. Hence Newton's method will not be very satisfactory near a multiple root. The point at infinity is a repelling fixed point with multiplier $d/(d-1)$, where d is the degree of the polynomial $g(z)$. This means that large values of z_n will tend to be pushed away from infinity.

If $g(z)$ has a root of order k at the point w, then $g(z)/g'(z)$ has a simple root at w. If we apply the standard Newton's method to $g(z)/g'(z)$ we obtain

$$M(z) = z - \frac{g(z)g'(z)}{[g'(z)]^2 - g(z)g''(z)}.$$

This is called *Newton's method for multiple roots* [4, Eqs. 8.6–24]. It is quadratically convergent at every root of $g(z)$, but could be more complicated to compute than the standard Newton's method because it involves second derivatives. Furthermore, there may be fixed points of $M(z)$ that are not roots of $g(z)$. The critical points of $g(z)$, that are not also roots of $g(z)$, are extraneous fixed points for this Newton's method for multiple roots. However, if w is such an extraneous fixed point then the multiplier is $M'(w) = 2$, which means that w is a repelling fixed point and so will not affect Newton's method. If $g(z)$ is a polynomial, then infinity is never a fixed point for $M(z)$.

Figure 1 shows the basins of attraction of the roots when this method is applied to the polynomial $g(z) = z^2(z^3 - 1)$, that has one double root and three simple roots. If the initial value z_0 lies in the white region then Newton's method for multiple roots will converge to within 0.001 of the origin in 10 iterations. Initial values lying in the colored regions will converge to one of the cube roots of unity, while those lying in the black regions have not converged to within 0.001 of any root in 10 iterations.

In practice, because of experimental errors or truncation of coefficients, the polynomial may not have a multiple root, but two or more roots that are very close together. This causes no problem to this Newton's method for multiple roots. For example, if the double root in the polynomial of Fig. 1 was split into two close roots to obtain $g(z) = z(z + 0.01)(z^3 - 1)$, then the rate of convergence of Newton's method for multiple roots is unchanged. Notice that the basins of attraction of the three cube roots in Figs. 1 and 2, which are the red, blue, and green areas, are basically the same. The white region in Fig. 1, which is the basin of attraction of the double root at the origin, is the union of the

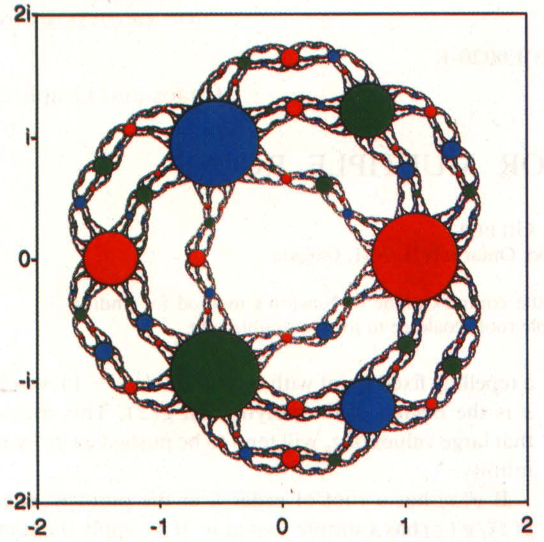

Fig. 1. Newton's Method for multiple roots applied to $g(z) = z^2(z^3 - 1)$. After 10 iterations points in the white and colored regions come to within 0.001 of a root.

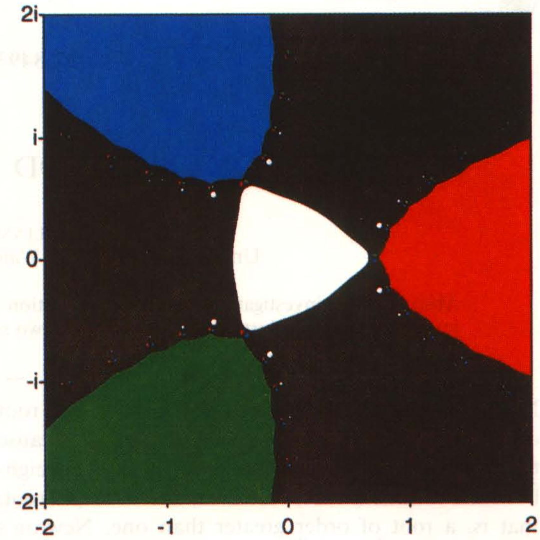

Fig. 3. The standard Newton's Method applied to $g(z) = z^2(z^3 - 1)$. After 10 iterations points in the white and colored regions come to within 0.001 of a root while points in the black region have not come close to any root.

white and yellow regions of Fig. 2, which are the basins of attraction of the two close roots, 0 and -0.01.

3. OTHER METHODS FOR A MULTIPLE ROOT

Let us compare the above method with other Newton methods for multiple roots. It is possible to use the standard Newton's method. Figure 3 illustrates the basins of attraction of the roots of $g(z) = z^2(z^3 - 1)$ after 10 iterations. If this figure is compared to Fig. 1, we see that there is a large black area in the basin of attraction of the double root 0, where points have not come close to the root after 10 iterations, because the method only converges linearly at the double root. Even if we look at the standard Newton's method applied to the modified polynomial $g(z) = z(z + 0.01)(z^3 - 1)$, that has no double roots, we see in Fig. 4 that the two close roots cause problems with Newton's method. This is because the range in which quadratic convergence occurs is extremely small near these close roots.

Another technique for dealing with multiple roots is the *relaxed Newton's method*. If w is a root of $g(z)$ of order k then

$$N_k(z) = z - \frac{kg(z)}{g'(z)}$$

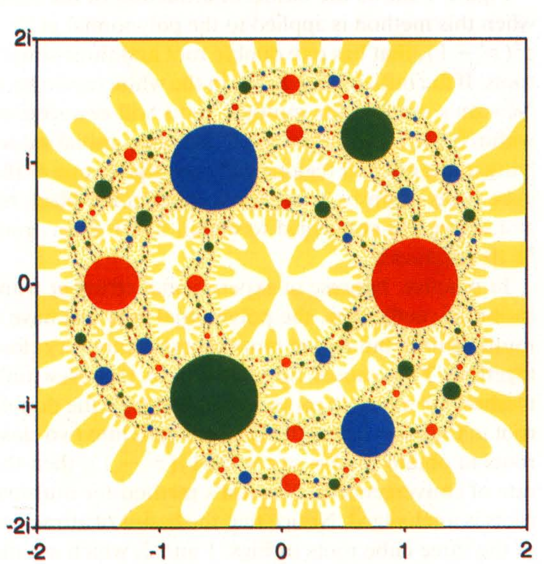

Fig. 2. Newton's Method for multiple roots applied to $g(z) = z(z + 0.01)(z^3 - 1)$. After 20 iterations points in the white and colored regions come to within 0.001 of a root.

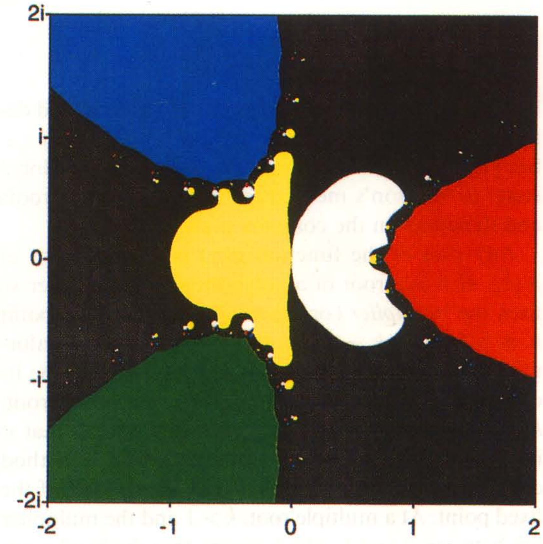

Fig. 4. The standard Newton's Method applied to $g(z) = z(z + 0.01)(z^3 - 1)$. After 10 iterations points in the white and colored regions come to within 0.001 of a root while points in the black region have not come close to any root.

will converge quadratically at the root w [4, Eqs. 8.6–13]. However, this method is hopeless if the root is not of order exactly k. The white region in Fig. 5 is the basin of attraction of the double root at the origin, under the relaxed Newton's method with $k = 2$. The cube roots of unity are fixed points under this method but $N_2'(w) = -1$, for any cube root w. Since this multiplier has modulus 1, a simple root of $g(z)$ is a neutral (or indifferent) fixed point for the method $N_2(z)$. Newton's method will behave very badly near these roots. Any neighborhood of such a neutral fixed point contains some points that are in the basin of attraction of the neutral point and some points that are not. The black regions in Fig. 5 are the basins of attraction of the cube roots, but it would require thousands of iterations for points in these regions to come close to the roots[5].

If some roots of $g(z)$ are only approximately multiple, then the convergence of the relaxed Newton's method at these roots changes drastically. Figure 6 illustrates the relaxed Newton's method $N_2(z)$ applied to $g(z) = z(z + 0.01)(z^3 - 1)$ after 3000 iterations. Compare Fig. 5 and Fig. 6. The large white area in Fig. 5, which is the basin of attraction of the double root, now contains large areas of black in Fig. 6 that have not converged to a root after 3000 iterations! However this is not a good representation of the actual basins of attractions of the two close roots. This diagram is very sensitive to how close we require the method to come to a root; changing the tolerance 0.004 to another value would substantially alter the diagram. In contrast, Newton's method for a multiple root is insensitive to the tolerance parameter; changing the tolerance from 0.001 in Figs. 1 or 2 would not change the figures significantly. Notice also that the values in the black regions of Fig. 5 are now colored in Fig. 6, indicating that they have converged to the cube roots of unity. This would have also occurred in Fig. 5, if we had iterated the method 3000 times.

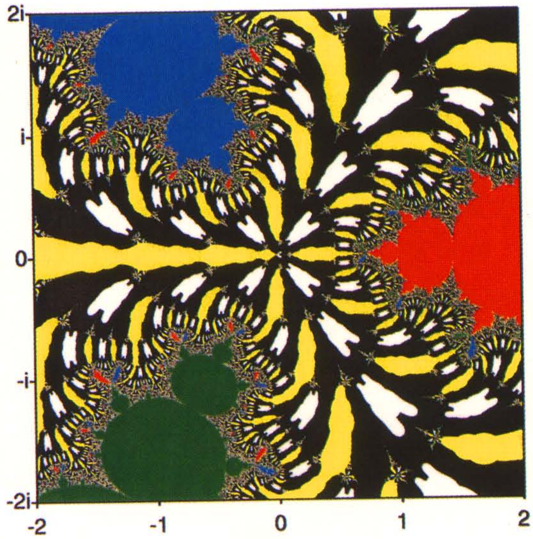

Fig. 6. The relaxed Newton's Method for a double root applied to $g(z) = z(z + 0.01)(z^3 - 1)$. After 3000 iterations points in the white and colored regions come to within 0.004 of a root.

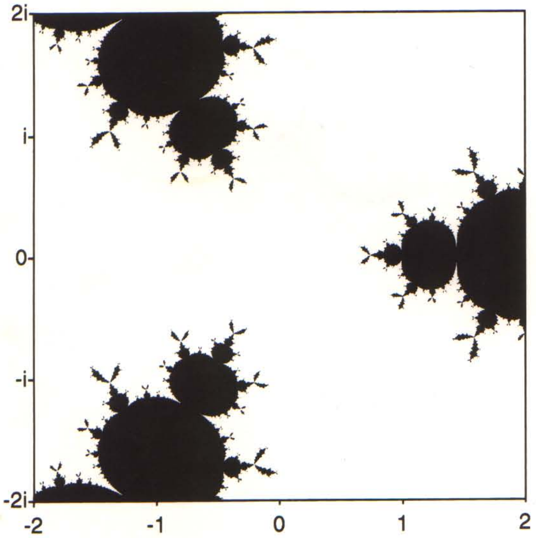

Fig. 5. The relaxed Newton's Method for a double root applied to $g(z) = z^2(z^3 - 1)$. After 100 iterations points in the white region come to within 0.001 of the double root at the origin while points in the black regions have not come close to any root.

Figures 1, 3, and 5 show three different Newton methods applied to the same polynomial with a double root. All the roots in the three figures are the same, but the basins of attractions are quite different, and the rate of convergence to the roots also varies tremendously. Likewise, Figs. 2, 4, and 6 show the three different Newton methods applied to another polynomial that has two almost equal roots. These figures show how sensitive the methods are to a double root dividing into two simple roots.

These last two figures show that the relaxed Newton's method is very sensitive to the two roots being exactly equal, and so it will never be a good method in practice, even if the initial approximation is quite close to the (nearly) equal roots. Newton's method for a multiple root is much safer.

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