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# THE COMPLEX DYNAMICS OF NEWTON'S METHOD FOR A DOUBLE ROOT

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**Abstract**—We show that the relaxed Newton's method for finding the roots of a cubic with one double root is conjugate, by a linear fractional transformation on the Riemann sphere, to the iterations of the quadratic $p(z) = z^2 - 0.75$.

## 1. INTRODUCTION

Newton's Method for approximating the real or complex roots of a polynomial $g(z)$ consists of iterating the function

$$N(z) = z - \frac{g(z)}{g'(z)}.$$

By starting with an initial approximation, $z_0$, sufficiently close to a root of $g$, the sequence of iterates, $z_{k+1} = N(z_k)$, will converge to the root. If the root is simple, the sequence converges quadratically, but if the root is a multiple one the convergence is only linear [1]. Newton's method can be modified to improve the convergence to multiple roots. If $g(z)$ is known to have a multiple root of order exactly $m$, then apply Newton's method to $\sqrt[m]{g(z)}$ to obtain

$$N_m(z) = z - \frac{g(z)^{\frac{1}{m}}}{\frac{1}{m}g(z)^{\frac{1}{m}-1}g'(z)} = z - \frac{mg(z)}{g'(z)}. \tag{1}$$

This is called *Newton's method for a root of order m* or the *relaxed Newton's method.* This relaxed Newton's method will converge quadratically to a root of order $m$.

For each polynomial $g(x)$, Newton's method defines a dynamical system on the complex Riemann sphere. In [1], Haeseler and Peitgen discuss the basins of attraction of the roots and give an overview of the complex dynamics of Newton's method for a rational function. We investigate the basins of attraction of the roots for the relaxed Newton's method. We give a complete description for the relaxed Newton's method for a double root applied to a cubic with a double root, and show that the dynamics is conjugate to that of a well-known Julia set. There is a similar result for the relaxed Newton's method for a root of order $m$ applied to a polynomial of degree $m + 1$. We also show that the Julia sets, obtained by Benzinger, Burns, and Palmore [2] in applying the standard Newton's method to the family of functions $(z - 1)(z + \alpha)^\alpha$, are conjugate to the Julia sets of quadratics.

## 2. RELAXED NEWTON'S METHOD FOR A DOUBLE ROOT

THEOREM 1. *The relaxed Newton's method, $N_2$ for a double root, applied to any cubic equation with a double root is conjugate, by a linear fractional transformation on the Riemann sphere, to the iterations of the quadratic $p(z) = z^2 - 3/4$.*

PROOF.

Let the cubic be $g(z) = (z-a)^2(z-b)$, where $a$ and $b$ are distinct complex numbers. Newton's method for a double root applied to this cubic is

$$N_2(z) = z - \frac{2g(z)}{g'(z)} = \frac{z^2 + az - 2ab}{3z - a - 2b}$$

and its derivative is

$$N_2'(z) = \frac{(z-a)(3z + a - 4b)}{(3z - a - 2b)^2}.$$

The fixed points of $N_2$ are $z = a$ and $z = b$. Since $N_2'(a) = 0$, $a$ is a superattractive fixed point and, since $N_2'(b) = -1$, $b$ is a neutral fixed point. The critical points of $N_2$ are $z = a$ and $z = (4b - a)/3$. The critical points of the quadratic $p(z) = z^2 + c$ on the Riemann sphere are $z = 0$ and $z = \infty$, while $z = \infty$ is also a superattractive fixed point.

The linear fractional transformation (or Möbius transformation)

$$h(z) = \frac{3z + a - 4b}{2(z-a)}$$

sends $a$ to $\infty$ and $(4b - a)/3$ to 0, and so maps the critical points of $N_2$ to those of the quadratic $p$. Conjugate the map $N_2$, by the transformation $h$, to obtain the quadratic

$$p(z) = h \circ N_2 \circ h^{-1}(z) = z^2 - \frac{3}{4}.$$

The constant factor in $h$ was chosen so that the coefficient of $z^2$ became 1.

Hence the dynamics of the relaxed Newton's method $N_2$ is conjugate to the dynamics of $p$, under the map $h$ on the Riemann sphere. ∎

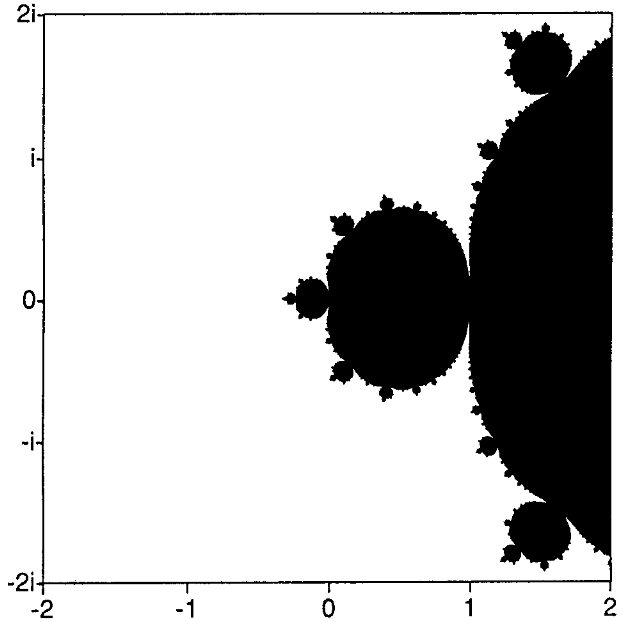


Figure 1. The white area is the basin of attraction, after 100 iterations, of the double root $-1$, when $N_2$ is applied to $(z+1)^2z$.

Figure 1 shows the basin of attraction for the double root in the complex plane when the relaxed Newton's method $N_2$ is applied to the cubic $g(z) = z(z+1)^2$. All initial values in the white region will converge to the double root $a = -1$, while the initial values in the black region do not converge to within 0.001 of a root in 100 iterations. The conjugate quadratic $p(z) = z^2 - 3/4$

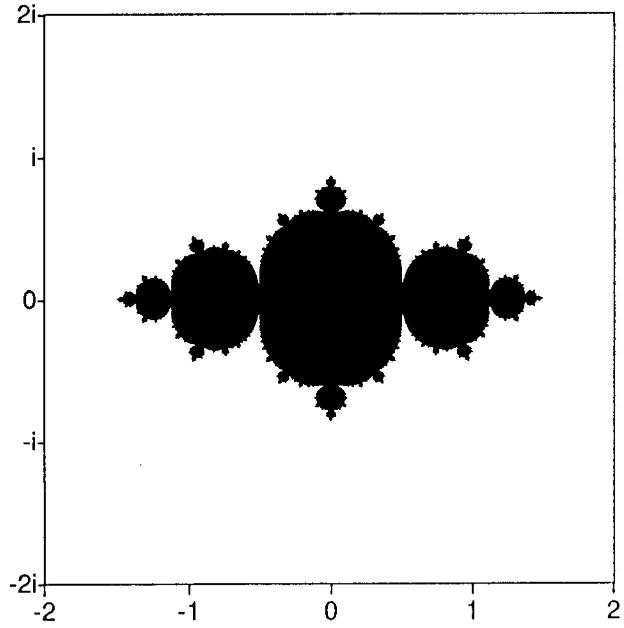


Figure 2. The filled-in Julia set for $p(z) = z^2 - 0.75$.

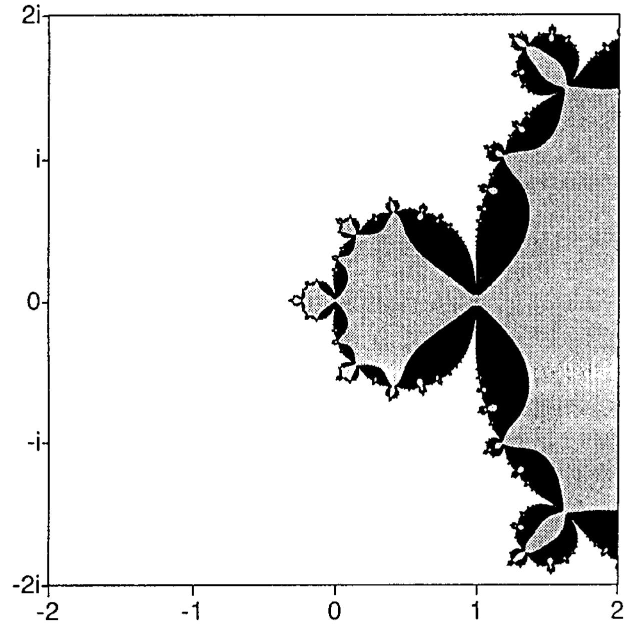


Figure 3. The same as Figure 1, except after 1195 iterations points in the gray area come to within 0.01 of the simple root 0.

gives rise to the filled-in Julia set shown in Figure 2. The actual Julia set is the boundary of the black region. The extreme left of the black region in Figure 1 is $(a + 2b)/3 = 1/3$ and this gets mapped, under $h$, to $-3/2$, which is the extreme left of the Julia set in Figure 2. The extreme right of the Julia set, $3/2$, comes from the point at infinity in Figure 1.

The quadratic $p$ corresponds to the point $3/4$ in the Mandelbrot set that is at the intersection of the cardioid and the largest circle. The dynamics of this quadratic are well-known. There is a repulsive fixed point at $3/2$ at the extreme right of the Julia set, and there is a neutral fixed point at $1/2$, that is the image of the simple root $b$. The map $p(z) = z^2\, 3/4$ is also conjugate to the map $z \to -z + z^2$, shown in [3] that has the neutral fixed point at the origin. By the Flower Theorem [3,4], there are petals inside the black region, in the neighborhood of the neutral fixed

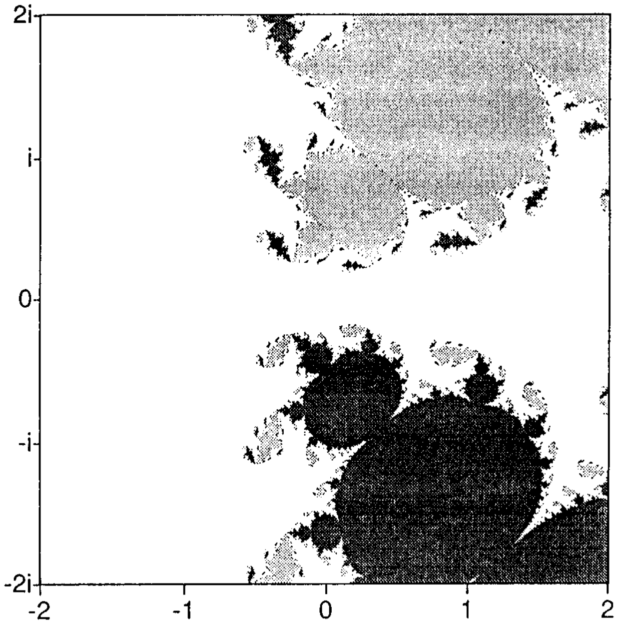


Figure 4. The basins of attraction for $N_2$ applied to $(z+1)^2(z^2+0.25)$. After 1000 iterations the points shaded white, light gray and dark gray, come to within 0.01 of the roots $-1$, $i/2$ and $-i/2$, respectively.

point, that converge to that fixed point. However the Julia set contains a sequence of repelling periodic points that converge to the neutral fixed point.

Hence the points inside the black region in Figure 1 do converge to the root $b = 0$, but do so extremely slowly, as shown in Figure 3. However, there is no complete neighborhood of $b$ in which points all converge to $b$, since any such neighborhood intersects the Julia set. Points starting on the the Julia set will not converge, but will remain on the boundary under iterations of $N_2$.

Benzinger, Burns, and Palmore [2] discuss Newton's method for the family of functions

$$f_\alpha(z) = (z + \alpha)^\alpha (z - 1). \tag{2}$$

For $\alpha = 1/2$, this gives the function $(z-1)\sqrt{z+0.5}$ and the basins of attraction for Newton's method are shown in [2]. This is the same as our Figure 1, since it follows from equation (1) that the relaxed Newton's method, $N_2$, for $(z-1)^2(z+0.5)$ is the same as Newton's method, $N$, for $(z-1)\sqrt{z+0.5}$.

## 3. RELAXED NEWTON'S METHOD FOR A ROOT OF ORDER $M$

To investigate more generally what happens to the relaxed Newton's method, apply $N_m$ to the function $g(z) = q(z)(z-b)^k$. Assume that $b$ is not a root of $q$ and assume that $q$ has finite derivatives. The root $b$ has order $k$ and

$$N_m'(b) = 1 - \frac{m}{k}$$

so that $b$ is a superattractive fixed point only if $k = m$. Otherwise, if $k < m/2$, $|N_m'(b)| > 1$ and $b$ is a repulsive fixed point. If $k = m/2$, $|N_m'(b)| = 1$ and $b$ is a neutral fixed point, as the cubic above shows. If $k > m/2$, $|N_m'(b)| < 1$ and $b$ is an attractive fixed point for $N_m$, but the convergence is only linear, not quadratic.

In the typical situation where we apply the relaxed Newton's method, $N_2$, to a function with one double root and no other multiple roots, all the simple roots will be neutral fixed points. For example, Figure 4 shows the basins of attraction for a quartic with one double root. Theorem 1 and its proof can be easily generalized to the relaxed Newton's method $N_m$.

THEOREM 2. *The relaxed Newton's method, $N_m$, applied to any polynomial of degree $(m+1)$ with a root of order $m$, is conjugate, by a linear fractional transformation on the Riemann sphere, to the iterations of the quadratic*

$$p(z) = z^2 + \frac{1-m^2}{4}.$$

In particular, when $m = 3$, the Julia set of $z^2 - 2$ is the line segment on the real axis between $-2$ and 2. Therefore the basin of attraction of the triple root of $(z-a)^3(z-b)$ under $N_3$ is the whole of the complex plane, except for a straight cut from $(a+3b)/4$ through $b$ to $\infty$. For higher values of $m$, the Julia set is totally disconnected. Since the relaxed Newton's method applied to $g(z)$ is the same as the standard Newton's method applied to $\sqrt[m]{g(z)}$, Theorem 2 can be reformulated as follows for $\alpha$ a positive real number.

THEOREM 3. *The standard Newton's method, $N$, applied to $(z-a)(z-b)^\alpha$, is conjugate, by a linear fractional transformation on the Riemann sphere, to the iterations of the quadratic*

$$p(z) = z^2 + \frac{1}{4} - \frac{1}{4\alpha^2}.$$

This theorem explains the Julia sets obtained by Benzinger, Burns, and Palmore [2], in applying Newton's method to the functions in equation (2). The Julia sets of $f_\alpha$, for $1/3 < \alpha \leq \infty$, are connected since they are conjugate to the Julia sets of the quadratics $p(z) = z^2 + c$ for $-2 < c \leq 1/4$.

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