H-SPACES AND WEAK COCATEGORY

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1. Introduction

THE object of this paper is to investigate the duals, in the sense of Eckmann–Hilton, of the Lusternik–Schnirelmann category and related invariants.

In his survey of numerical homotopy invariants Ganea [in (**5**) 470] pointed out that the dual of the weak category had not yet been defined. Since then Ganea (**6**) has given a definition of cocategory which dualizes correctly and allows the notion of weak cocategory to be defined. As usual, the spaces of cocategory 1 are the non-contractible H-spaces.

Starting from Ganea's definition we prove the following theorem, generalizing Theorem 2.4 of (**5**), which shows the relations between the invariants and also shows that they are all different.

THEOREM 1.1. *Let A be a connected countable CW-complex. Then*

$$\operatorname{cocat} A \geqslant \operatorname{wcocat} A \geqslant \operatorname{nil} A \geqslant \text{W-long} A$$

and furthermore all the inequalities can occur.

All the definitions and proofs of the inequalities will be given in § 2. Examples 2.5, 2.6, and 3.5, which are all spaces with two non-zero homotopy groups, show that the inequalities may be strict.

If $\Omega^* A$ is the space of free loops on A, the natural fibration

$$\mathscr{A}: \Omega A \xrightarrow{g} \Omega^* A \xrightarrow{f} A$$

always has a retraction $\rho: \Omega^* A \to \Omega A$ such that $\rho \circ g \simeq 1$ when A is an H-space. By using the space in Example 3.5 we show in § 4 that the existence of a retraction is not a sufficient condition for A to be an H-space. This answers a question of I. M. James.

This paper is part of a doctoral dissertation written under the supervision of Dr. I. M. James at Oxford University. I would also like to thank Professor T. Ganea for his helpful advice. In particular Proposition 4.2 is due to him.

2. Definitions and examples

All the spaces we will consider have the homotopy type of connected countable CW-complexes and have a base point denoted by $*$. All the maps will preserve base points unless the contrary is explicitly stated. The constant map is denoted by 0 and the identity map by 1. The set of homotopy classes of maps from X to Y is denoted by $[X, Y]$ and we will not usually distinguish between a map, its homotopy class, or its corresponding cohomology class.

Following § 6 of (**6**), for any space A construct the ladder of cofibrations

$$\mathscr{C}_k\colon A \xrightarrow{e_k} F_k \xrightarrow{f_k} B_k \quad (k \geqslant 0).$$

Let $\mathscr{C}_0$ be the standard cofibration in which $F_0 = CA$, the reduced cone over A, and $B_0 = \Sigma A$, the reduced suspension of A. Suppose inductively that $\mathscr{C}_k$ has been defined. Let F'_{k+1} be the fibre of f_k; since $f_k \circ e_k \simeq 0$ we can lift e_k to a map $e'_{k+1}\colon A \to F'_{k+1}$. Convert e'_{k+1} into a cofibre map $e_{k+1}\colon A \to F_{k+1}$ where F_{k+1} is the reduced mapping cylinder of e'_{k+1}. Let B_{k+1} be the cofibre of e_{k+1} and $f_{k+1}\colon F_{k+1} \to B_{k+1}$ the identification map.

When e_k is converted into a fibration let the fibre be D_k with projection $d_k\colon D_k \to A$.

Since e_1 is homotopic to the natural embedding $A \to \Omega\Sigma A$, the first two rungs of the ladder are homotopic to the following diagram.

$$\begin{array}{ccccccc}
A & \xrightarrow{d_0} & A & \xrightarrow{e_0} & CA & \xrightarrow{f_0} & \Sigma A \\
 & & \| & & \uparrow & & \\
D_1 & \xrightarrow{d_1} & A & \xrightarrow{e_1} & \Omega\Sigma A & \xrightarrow{f_1} & B_1
\end{array}$$

Definition 2.1. The *cocategory* of A, $\operatorname{cocat} A$, is the least integer $k \geqslant 0$ for which there exists a map $r\colon F_k \to A$ such that $r \circ e_k \simeq 1$; if no such integer exists $\operatorname{cocat} A = \infty$.

Definition 2.2. The *weak cocategory* of A, $\operatorname{wcocat} A$, is the least integer $k \geqslant 0$ for which $d_k \simeq 0$; if no such integer exists $\operatorname{wcocat} A = \infty$.

With this definition the dual invariant weak category is slightly different from that originally defined by Berstein and Hilton [see (**7**)].

The spaces of cocategory 0 and weak cocategory 0 are the contractible spaces. It follows from Theorem 1.8 of (**9**) that the spaces of cocategory 1 are the non-contractible H-spaces.

Let $(\Omega A)^{(k+1)}$ be the $(k+1)$-fold smash product of ΩA and let

$$\phi_k\colon (\Omega A)^{(k+1)} \to \Omega A$$

be the commutator map with respect to the loop space multiplication on ΩA as defined in § 6 of (**6**).

Definition 2.3. The *nilpotency class of the loop space* of A, $\operatorname{nil} A$, is the least integer $k \geqslant 0$ for which $\phi_k \simeq 0$; if no such integer exists $\operatorname{nil} A = \infty$.

Recall that $\text{nil}\, A = \sup \text{nil}[X, \Omega A]$ where X ranges over all based spaces [(**1**) Theorem 2.7].

Definition 2.4. The *Whitehead product length* of A, W-long A, is the least integer $k \geqslant 0$ such that $[\alpha_1, \ldots, [\alpha_k, \alpha_{k+1}] \ldots] = 0$ for all $\alpha_i \in \pi_*(A)$ in positive dimensions; if no such integer exists W-long $A = \infty$.

It is clear from the definitions that for any space A, $\text{cocat}\, A \geqslant \text{wcocat}\, A$. The other two inequalities in Theorem 1.1 follow from (**6**) Lemma 6.4 and (**1**) Theorem 4.6.

EXAMPLE 2.5. *Let A be defined by the 2-stage Postnikov system*

$$K(\mathbf{Z}, 5) \to A \to K(\mathbf{Z}, 2)$$

where the k-invariant is $u^3 \in H^6(\mathbf{Z}, 2; \mathbf{Z})$, u being the fundamental class in $H^2(\mathbf{Z}, 2; \mathbf{Z})$. Then $\text{wcocat}\, A = 2$ *and* $\text{nil}\, A = 1$.

Proof. This space A is obtained from the complex projective plane $\mathbf{C}P^2$ by killing off the homotopy groups in dimensions greater than 5. Berstein and Ganea in (**1**) 3.10 show that $\text{nil}\, A = 1$.

By (**14**) Corollary 2 there is a non-trivial higher-order Whitehead product in A, $[\alpha, \alpha, \alpha] = \pm 3!\, \beta$ where α generates $\pi_2(A) = \mathbf{Z}$ and β generates $\pi_5(A) = \mathbf{Z}$. Consider the homotopy exact sequence obtained by converting $e_1 \colon A \to \Omega\Sigma A$ into a fibration:

$$\pi_5(D_1) \xrightarrow{d_{1*}} \pi_5(A) \xrightarrow{e_{1*}} \pi_5(\Omega\Sigma A).$$

By the naturality of the higher-order Whitehead product,

$$e_{1*}[\alpha, \alpha, \alpha] = [e_{1*}\alpha, e_{1*}\alpha, e_{1*}\alpha] = 0$$

since all Whitehead products vanish in the H-space $\Omega\Sigma A$. Hence by exactness there exists $\gamma \in \pi_5(D_1)$ such that $d_{1*}\gamma = [\alpha, \alpha, \alpha]$ and in particular $d_1 \not\simeq 0$. Therefore by the definition $\text{wcocat}\, A > 1$.

Since A is a 2-stage Postnikov system, it follows from (**6**) Lemma 6.3 that $\text{cocat}\, A = \text{wcocat}\, A = 2$.

EXAMPLE 2.6. *Let A be defined by the 2-stage Postnikov system*

$$K(\mathbf{Z}_2, 4) \to A \to K(\mathbf{Z}_2, 2)$$

where the k-invariant is $u . Sq^1 u \in H^5(\mathbf{Z}_2, 2; \mathbf{Z}_2)$, u being the fundamental class in $H^2(\mathbf{Z}_2, 2; \mathbf{Z}_2)$. Then $\text{nil}\, A = 2$ *and* W-long $A = 1$.

Proof. The space A has trivial homotopy groups except in dimensions 2 and 4. Thus all Whitehead products vanish and W-long $A = 1$.

The k-invariant of A is non-primitive. Hence, by (**3**) Theorem 6, A is not an H-space and $\text{cocat}\, A = 2$. But it follows from Sugawara's

Theorem [(**15**) Theorem 8.1] that ΩA is not homotopy commutative and so $\operatorname{nil} A = 2$.

3. Weak cocategory

Throughout this section we will take the space A as lying in the fibration sequence

$$\Omega X \xrightarrow{\Omega k} \Omega K \xrightarrow{m} A \xrightarrow{l} X \xrightarrow{k} K$$

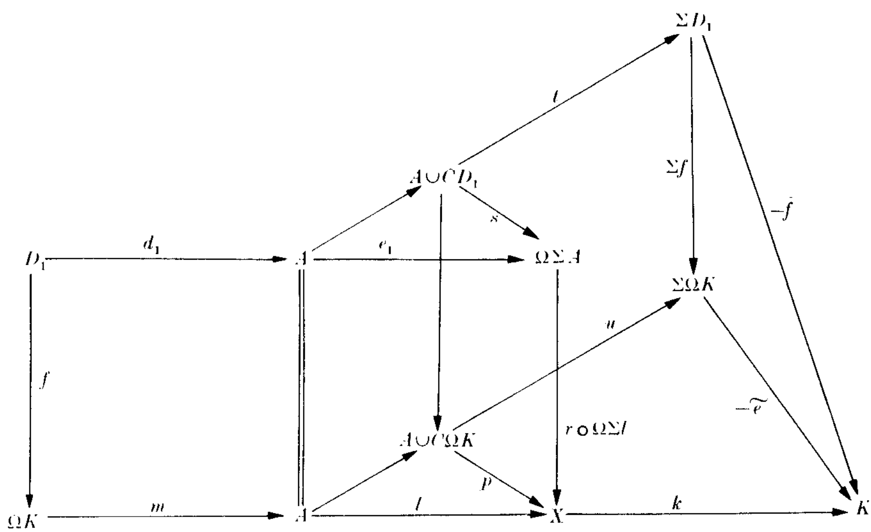
where X is an H-space with multiplication μ, $K = K(\pi, n)$ is an Eilenberg–MacLane space, and $k \in H^n(X; \pi)$ is the k-invariant. This is the general situation when A is the $(r+1)$th stage of a Postnikov system and there is a multiplication structure up to the rth stage X.

In this section we find a condition for $\operatorname{wcocat} A \leqslant 1$ and use it to give an example of a space A with $\operatorname{wcocat} A = 1$ but which is not an H-space.

The multiplication μ defines a retraction $r: \Omega\Sigma X \to X$ [see (**9**) Theorem 1.8] of the embedding $e_1': X \to \Omega\Sigma X$. Then

$$\begin{aligned} r \circ \Omega\Sigma l \circ e_1 &\simeq r \circ e_1' \circ l \quad \text{(by naturality)} \\ &\simeq l. \end{aligned}$$

This homotopy induces a map $f: D_1 \to \Omega K$ between the fibres of e_1 and l.



(3.1)

Let σ denote the cohomology suspension of the fibration

$$D_1 \to A \to \Omega\Sigma A.$$

THEOREM 3.2. *wcocat $A \leqslant 1$ if and only if there exists a map $w\colon D_1 \to \Omega X$ such that*

$$\Omega k \circ w = \sigma(k \circ r \circ \Omega\Sigma l) \in H^{n-1}(D;\pi).$$

This will follow from the next proposition and from the homotopy exact sequence of the fibration $A \to X \to K$.

PROPOSITION 3.3. $f = \sigma(k \circ r \circ \Omega\Sigma l)$.

Proof. In diagram 3.1 $\tilde{e}\colon \Sigma\Omega K \to K$ is the evaluation map and $\bar{f}\colon \Sigma D_1 \to K$ is the adjoint of f. The maps p and s come naturally from extending the fibrations l and e_1 [see § 1 of (**6**)]. It follows from (**13**) Lemma 14 that $k \circ p \simeq -\tilde{e} \circ u$ and hence by the naturality of the cofibration sequences of m and d_1 the diagram 3.1 is homotopy commutative.

$$\begin{array}{ccccc}
H^n(\Omega\Sigma A;\pi) & \xrightarrow{\hat{e}_1^*} & H^n(A,D_1;\pi) & \xleftarrow{\delta} & H^{n-1}(D_1;\pi) \\
\updownarrow & & \updownarrow & & \updownarrow \\
 & & H^n(A \cup CD_1;\pi) & & [D_1,\Omega K] \\
 & & \updownarrow & & \updownarrow \\
[\Omega\Sigma A, K] & \xrightarrow{s^*} & [A \cup CD_1, K] & \xleftarrow{-t^*} & [\Sigma D_1, K]
\end{array} \qquad (3.4)$$

In diagram 3.4 all the vertical maps are isomorphisms, δ is the boundary homomorphism, and $\hat{e}_1^*$ is induced from e_1. The left-hand square is commutative by naturality. The commutativity of the right-hand square follows from the hexagonal lemma (**4**) I, 15.1 and the isomorphisms

$$[\Sigma D_1, K] \approx H^n(\Sigma D_1, \pi) \approx H^n(A \cup CD_1, A;\pi).$$

The cohomology suspension σ is determined by $\delta^{-1}\hat{e}_1^*$ and hence also by $-t^{*-1}s^*$. Now

$$\begin{aligned}
s^*(k \circ r \circ \Omega\Sigma l) &= k \circ r \circ \Omega\Sigma l \circ s \\
&= -\bar{f} \circ t \quad \text{(from diagram 3.1)} \\
&= -t^*(\bar{f}).
\end{aligned}$$

Hence, from diagram 3.4,

$$\delta(f) = \hat{e}_1^*(k \circ r \circ \Omega\Sigma l).$$

The suspension σ is determined modulo the image of d^*. But $d^* = 0$ since e_1^* is epimorphic in cohomology and hence $f = \sigma(k \circ r \circ \Omega\Sigma l)$.

EXAMPLE 3.5. *Let A be defined by the 2-stage Postnikov system*

$$K(\mathbf{Z}_4, 7) \to A \to K(\mathbf{Z}, 2)$$

where the k-invariant is $u^4 \in H^8(\mathbf{Z}, 2; \mathbf{Z}_4)$, *u being the fundamental class in* $H^2(\mathbf{Z}, 2; \mathbf{Z}_4)$. *Then* $\operatorname{cocat} A = 2$ *and* $\operatorname{wcocat} A = 1$.

Proof. There is, up to homotopy, only one multiplication μ on $K(\mathbf{Z}, 2)$ and

$$\mu^*(u^4) = 1 \otimes u^4 + 2u^2 \otimes u^2 + u^4 \otimes 1 \pmod 4.$$

Hence the k-invariant is non-primitive and, by (**3**) Theorem 6, A is not an H-space. Since A is a 2-stage Postnikov system, $\operatorname{cocat} A = 2$.

Since $\Omega k = 0$, by Theorem 3.2 $\operatorname{wcocat} A = 1$ if and only if

$$\sigma(k \circ r \circ \Omega\Sigma l) = 0 \in H^7(D_1; \mathbf{Z}_4).$$

In order to calculate $k \circ r$ we need to know the cup product in $H^*(\Omega\Sigma K(\mathbf{Z}, 2); \mathbf{Z}_4)$.

The space $X = K(\mathbf{Z}, 2)$ is homotopic to the infinite complex projective space and so its integer cohomology ring is $\mathbf{Z}[v]$ where $v \in H^2(X)$. Let $e_i \in H_{2i}(X)$ be dual to v^i. Then the diagonal map

$$\Delta_*: H_*(X) \to H_*(X) \otimes H_*(X)$$

is given by $\Delta_*(e_i) = \sum_{k=0}^{i} e_k \otimes e_{i-k}$.

Since $H_*(X)$ is free, by the Bott–Samelson theorem (**2**), $H_*(\Omega\Sigma X)$ is the free associative algebra generated by $\tilde{H}_*(X)$ with e_0 as unit. Denote elements of $H_*(\Omega\Sigma X)$ by $(a_1|\ldots|a_n)$ where $a_i \in \tilde{H}_*(X)$. Since $\Delta: \Omega\Sigma X \to \Omega\Sigma X \times \Omega\Sigma X$ is an H-map, it follows from the Eilenberg–Zilber theorem that

$$\Delta_*(a_1|\ldots|a_n) = (\Delta_* a_1|\ldots|\Delta_* a_n)$$

where $(a_1 \otimes b_1 \mid a_2 \otimes b_2) = (a_1 \mid a_2) \otimes (b_1 \mid b_2)$.

Let $\langle \bar{a}_1|\ldots|\bar{a}_n\rangle \in H^*(\Omega\Sigma X)$ be dual to $(a_1|\ldots|a_n)$ where $\bar{e}_i = v^i$. Hence, by using Δ_* and duality, we calculate the following cup product in $H^*(\Omega\Sigma X)$.

$$\begin{aligned}\langle v\rangle^4 &= (\langle v^2\rangle + 2\langle v \mid v\rangle)^2\\ &= \langle v^4\rangle + 6\langle v^2 \mid v^2\rangle + 4\langle v \mid v^3\rangle + 4\langle v^3 \mid v\rangle + 12\langle v \mid v \mid v^2\rangle +\\ &\qquad + 12\langle v \mid v^2 \mid v\rangle + 12\langle v^2 \mid v \mid v\rangle + 24\langle v \mid v \mid v \mid v\rangle.\end{aligned}$$

Since $H^*(\Omega\Sigma X)$ is free, $H^*(\Omega\Sigma X; \mathbf{Z}_4) = H^*(\Omega\Sigma X) \otimes \mathbf{Z}_4$ and

$$r^*u \in H^2(\Omega\Sigma X; \mathbf{Z}_4)$$

is the reduction mod 4 of $\langle v\rangle$. So

$$r^*k = r^*u^4 = \langle u^4\rangle + 2\langle u^2 \mid u^2\rangle \in H^8(\Omega\Sigma X; \mathbf{Z}_4),$$

where $\langle u^{i_1}|\ldots|u^{i_n}\rangle$ is the reduction mod 4 of $\langle v^{i_1}|\ldots|v^{i_n}\rangle$.

Hence $(\Omega\Sigma l)^* r^* k^* = 2\langle u^2 \mid u^2\rangle \in H^8(\Omega\Sigma A; \mathbf{Z}_4)$ where we use the same notation for the elements of the cohomology of A as those of X in dimensions less than 9.

By Proposition 3.3,

$$f = \sigma 2\langle u^2 \mid u^2\rangle. \tag{3.6}$$

Now consider the cohomology spectral sequence (mod 4) of the fibration homotopic to $D_1 \to A \to \Omega\Sigma A$. We have

$$E_2^{p,q} = H^p(\Omega\Sigma A; H^q(D_1; \mathbf{Z}_4))$$

and E_∞ is associated to a filtration of $H^*(A; \mathbf{Z}_4)$. For $p < 8$,

$$H^p(\Omega\Sigma A) \approx H^p(\Omega\Sigma X)$$

and is free. Hence for $p < 8$ by the universal coefficient theorem $E_2^{p,q} \approx H^p(\Omega\Sigma A) \otimes H^q(D_1; \mathbf{Z}_4)$.

In this spectral sequence, all the elements are eventually killed off except $H^n(A; \mathbf{Z}_4) \subset E^{n,0}$. So there exists $x \in E_2^{0,3}$ of order 4 such that

$$d_4(x) = \langle u \mid u\rangle.$$

Then

$$\begin{aligned} d_4(\langle u \mid u\rangle \otimes x) &= \langle u \mid u\rangle^2 \\ &= \langle u^2 \mid u^2\rangle + 2\xi \quad \text{where } \xi \in E_4^{8,0}. \end{aligned}$$

So in $E_8^{8,0}$, $2\langle u^2 \mid u^2\rangle \equiv 0 \pmod 4$ and hence $2\langle u^2 \mid u^2\rangle$ is in the kernel of the cohomology suspension σ. Therefore by (3.6) $f = 0 \in H^7(D_1; \mathbf{Z}_4)$ and wcocat $A = 1$.

4. The space of free loops

For any space A, a free loop in A is a map $\lambda: I \to A$, which is not required to preserve base points, but which satisfies $\lambda 0 = \lambda 1$. Denote the space of free loops on A by $\Omega^* A$. The base point of $\Omega^* A$ is the constant loop at the base point of A. There is a natural fibration

$$\mathscr{A}: \Omega A \xrightarrow{g} \Omega^* A \xrightarrow{f} A,$$

where $f(\lambda) = \lambda 0$ and g is the inclusion map. It follows from (**12**) that $\Omega^* A$ and ΩA have the homotopy type of countable CW-complexes if A does.

PROPOSITION 4.1. *If A is an H-space which is a countable CW-complex then the fibration $\mathscr{A}$ has a retraction $\rho: \Omega^* A \to \Omega A$ such that $\rho \circ g \simeq 1$.*

Proof. By Lemma 6.4 of (**11**) there is a multiplication μ on A such that $\mu(a, *) = \mu(*, a) = a$ for all $a \in A$. The proposition then follows from (**10**) Theorem 2.7.

T. Ganea has pointed out that the result also holds when wcocat $A \leqslant 1$ and hence Example 3.5 shows that the existence of a retraction in $\mathscr{A}$ is not a sufficient condition for A to be an H-space.

PROPOSITION 4.2. *If A is a countable CW-complex such that* wcocat $A \leqslant 1$ *then the fibration $\mathscr{A}$ has a retraction $\rho\colon \Omega^* A \to \Omega A$ such that $\rho \circ g \simeq 1$.*

Proof. In the homotopy commutative diagram 4.3 there is by Proposition 4.1 a retraction ρ' such that $\rho' \circ g' \simeq 1$. Now wcocat $A \leqslant 1$ so $d_1 \simeq 0$,

$$\begin{array}{ccccc}
\Omega D_1 & & & & D_1 \\
\downarrow{\scriptstyle \Omega d_1} & & & & \downarrow{\scriptstyle d_1} \\
\Omega A & \xrightarrow{\;g\;} & \Omega^* A & \xrightarrow{\;f\;} & A \\
{\scriptstyle\tau}\uparrow\downarrow{\scriptstyle\Omega e} & & \downarrow{\scriptstyle\Omega^* e} & & \downarrow{\scriptstyle e} \\
\Omega^2\Sigma A & \underset{\rho'}{\overset{g'}{\rightleftarrows}} & \Omega^*\Omega\Sigma A & \xrightarrow{\;f'\;} & \Omega\Sigma A
\end{array} \tag{4.3}$$

and it follows from the fibration sequence that Ωe has a retraction τ such that $\tau \circ \Omega e \simeq 1$ [see (**8**) Lemma 2.3].

Define $$\rho = \tau \circ \rho' \circ \Omega^* e,$$

so that $$\rho \circ g \simeq \tau \circ \rho' \circ g' \circ \Omega e \simeq \tau \circ \Omega e \simeq 1.$$

COROLLARY 4.4. *Let A be the space defined in Example 3.5, then the fibration $\mathscr{A}$ has a retraction $\rho\colon \Omega^* A \to \Omega A$ such that $\rho \circ g \simeq 1$ but A is not an H-space.*

REFERENCES

1. I. Berstein and T. Ganea, 'Homotopical nilpotency', *Illinois J. Math.* 5 (1961) 99–130.
2. R. Bott and H. Samelson, 'On the Pontryagin product in spaces of paths', *Comment. Math. Helv.* 27 (1953) 320–37.
3. A. H. Copeland, 'On H-spaces with two non-trivial homotopy groups', *Proc. Amer. Math. Soc.* 8 (1957) 184–91.
4. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology* (Princeton, 1952).
5. T. Ganea, 'On some numerical homotopy invariants', *Proc. Int. Congress of Math.* (1962) 467–72.
6. —— 'A generalization of the homology and homotopy suspension', *Comment. Math. Helv.* 39 (1965) 295–322.
7. W. J. Gilbert, 'Some examples for weak category and conilpotency', *Illinois J. Math.* 12 (1968) 421–32.
8. P. J. Hilton, 'Nilpotency and H-spaces', *Topology*, 3 (1965) 161–76.
9. I. M. James, 'Reduced product spaces', *Ann. of Math.* 62 (1955) 170–97.

10. I. M. James and E. Thomas, 'An approach to the enumeration problem for non-stable vector bundles', *J. Math. Mech.* 14 (1965) 485–506.
11. I. M. James and J. H. C. Whitehead, 'The homotopy theory of sphere bundles over spheres. I', *Proc. London Math. Soc.* 4 (1954) 196–218.
12. J. Milnor, 'On spaces having the homotopy type of a CW-complex', *Trans. Amer. Math. Soc.* 90 (1959) 272–80.
13. Y. Nomura, 'On mapping sequences', *Nagoya Math. J.* 17 (1960) 111–45.
14. G. J. Porter, 'Higher order Whitehead products and Postnikov systems', *Illinois J. Math.* 11 (1967) 414–16.
15. M. Sugawara, 'On the homotopy commutativity of groups and loop spaces', *Mem. Coll. Sci. Kyoto*, A33 (1960) 257–69.

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