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HOMOTOPICAL NILPOTENCE OF THE SEVEN SPHERE

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ABSTRACT. We prove that the homotopical nilpotence of S^7 is 3, with respect to any of its 120 H-space multiplications.

The homotopical nilpotence of S^3 has been calculated by Porter [4] for the standard multiplication and by Arkowitz and Curjel [1] for all of its twelve H-space multiplications. Arkowitz and Curjel mention that their methods lead to results on the multiplications on S^7 but do not calculate its homotopical nilpotence. By modifying their arguments with the Samelson products we obtain the results on S^7 easily.

We will denote the collection of homotopy classes of basepoint preserving maps from A to B by $[A, B]$ and we will not distinguish notationally between a map and its homotopy class. The multiplication and inverse in the unit Cayley numbers induce the standard multiplication $m \in [S^7 \times S^7, S^7]$ and two sided homotopy inverse $\nu \in [S^7, S^7]$ on the space S^7. For the H-space (S^7, m, ν) we define a *commutator map* $\phi: S^7 \times S^7 \to S^7$ by $\phi(x, y) = (x \cdot y) \cdot (x^{-1} \cdot y^{-1})$ using the multiplication m and inverse ν. Recall that the Cayley multiplication is not associative but is diassociative, i.e. any two elements generate an associative subalgebra. We now make a choice in bracketing to define inductively the *k-fold commutator map* $\phi: (S^7)^k \to S^7$ by $\phi_k = \phi \circ (\phi_{k-1} \times 1)$ where $\phi_1 = 1$, the identity map on S^7. It is well known that ϕ_k induces a unique homotopy class $\psi_k \in [\bigwedge^k S^7, S^7]$ with $\psi_k \circ q_k = \phi_k$, where $\bigwedge^k S^7$ is the k-fold smash product of S^7 (homeomorphic to S^{7k}) and $q_k: (S^7)^k \to \bigwedge^k S^7$ is the projection map. The *homotopical nilpotence* of the H-space (S^7, m, ν) written $\operatorname{nil}(S^7, m, \nu)$, is the least integer k such that ϕ_{k+1} (and hence ψ_{k+1}) is nullhomotopic.

THEOREM. $\operatorname{nil}(S^7, m, \nu) = 3$.

PROOF. James [2, p.176] proves that ψ_2 generates $\pi_{14}(S^7) = Z_{120}$ so that in Toda's notation [5] its 2-component is σ', its 3-component is $\alpha_2(7)$ and its 5-component is $\alpha_1(7)$. Now

$$\psi_3 = \psi \circ (\psi \wedge 1) = \psi \circ \Sigma^7 \psi \in \pi_{21}(S^7) = Z_{24} \oplus Z_4$$

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and its 2-component is $\sigma' \circ \Sigma^7 \sigma' = 2\sigma' \circ \sigma_{14} \neq 0$ [5]. The element α_2 is defined in terms of a Toda bracket and so the 3-component of ψ_3 is

$$\begin{aligned}\alpha_2(7) \circ \alpha_2(14) &\in \{\alpha_1(7), 3\iota_{10}, \alpha_1(10)\} \circ \alpha_2(14)\\ &\subset \{\alpha_1(7), 3\iota_{10}, \alpha_1(10) \circ \alpha_2(13)\} = 0\end{aligned}$$

since $\alpha_1(10) \circ \alpha_2(13) = 0$ by Lemma 13.8 of [5]. Hence ψ_3 has only a 2-component and

$$\psi_4 = \psi_3 \circ \Sigma^{14}\psi \in \pi_{28}(S^7) = \boldsymbol{Z}_6 \oplus \boldsymbol{Z}_2 \quad \text{by [3]}$$

and so $\psi_4 = 4\sigma' \circ \sigma_{14} \circ \sigma_{21} = 0$ which proves the theorem.

There are 120 different homotopy classes of multiplications on S^7 and as in Lemma 2 of [1] it can be shown that they can be written additively in the form

$$m^{(t)} = m + t\phi \in [S^7 \times S^7, S^7], \qquad t = 0, 1, \cdots, 119.$$

Also as in Lemma 3 of [1], ν is a homotopy inverse for each of these multiplications.

COROLLARY. $\mathrm{nil}(S^7, m^{(t)}, \nu) = 3$ *for* $t = 0, 1, \cdots, 119$.

PROOF. Denote by $\psi_k^{(t)} \in [\bigwedge^k S^7, S^7]$ the r-fold smash commutator map defined on the H-space $(S^7, m^{(t)}, \nu)$. Then James [2, p. 176] and Arkowitz and Curjel [1, Lemma 4] prove that $\psi_k^{(t)} = (2t+1)\psi_k$. Hence $\psi_3^{(t)}$ is nonzero and $\psi_4^{(t)}$ is zero, which proves the corollary.

Changing the choice of bracketing in the definition of the k-fold commutator map will at most affect a sign change in ψ_k, so that the homotopy nilpotence is independent of the choice of bracketing.

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