CO781 / QIC 890:

Theory of Quantum Communication

Topic 5, part 4

Consequences of the LSD theorem

-- so what IS the quantum capacity of a quantum channel?

* what we know (degradable channels, e.g., erasure channel)

Today

-- bounds (continuity, 1-shot)

Thur

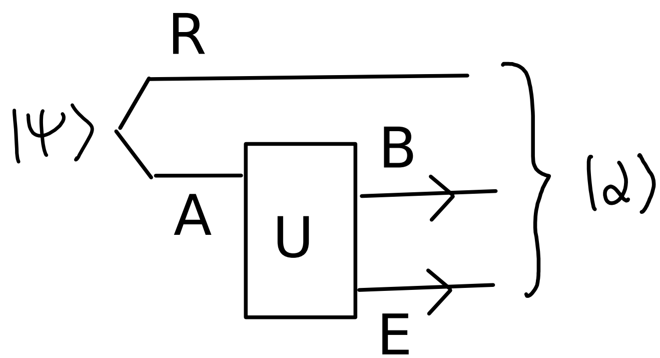
* what we know we don't know
  (nonadditivity of coherent info -- depolarizing channel)

Recall:

$$Q^{(1)}(N) := \max_{|\psi\rangle} I_c(R\rangle B)_{I\otimes N(|\psi\rangle\langle\psi|)_{RB}}$$

$$= \max_{|\psi\rangle} \left(S_B - S_{RB}\right)_{I\otimes N(|\psi\rangle\langle\psi|)_{RB}}$$

$$= \max_{|\psi\rangle} \left(S_B - S_E\right)_{|\alpha\rangle}$$



$$Q^{(r)}(N) := \frac{1}{r}\, Q^{(1)}(N^{\otimes r})$$

LSD thm: $Q(N) = \sup_r Q^{(r)}(N)$

How to evaluate the coherent information for any arbitrary channel?

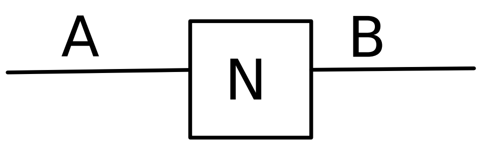
Example: erasure channel (on a qubit)

input space A (2-dim)
output space B1 (3-dim)

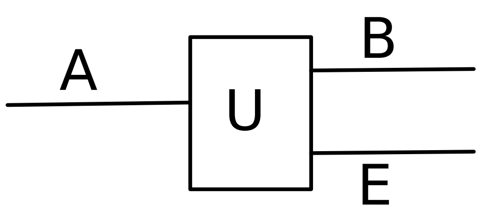
$$\mathcal{E}_p(\rho) = (1-p)\rho + p\,|e\rangle\langle e|$$

erasure prob error symbol ortho to all inputs

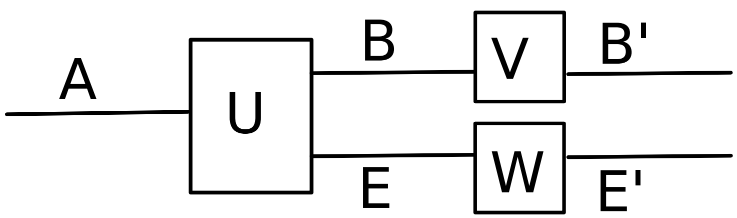
In general: each of the following is equivalent for the purpose of understanding channel coding and capacities:



(specified as a linear map from states on A to states on B)



(any isometric extension specified on a basis on A)



(U from above, V, W isometries)

Example: erasure channel (on a qubit)

input space A (2-dim)
output space B1 (3-dim)

$$\mathcal{E}_p(\rho) = (1-p)\,\rho + p\,|e\rangle\langle e|$$

erasure prob   error symbol ortho to all inputs

Consider the following isometry from B1 to B1 B2:

1. Attach $|0\rangle_{B2}$
2. Apply unitary $(|0\rangle\langle 0| + |1\rangle\langle 1|)_{B1} \otimes I_{B2} + |e\rangle\langle e|_{B1} \otimes \sigma_{x\,B2}$

i.e., with no erasure, Bob gets $|0\rangle_{B2}$, with erasure, Bob gets $|1\rangle_{B2}$.

$$\mathcal{E}_p^{\prime}(\rho) = (1-p)\,\rho_{B1} \otimes |0\rangle\langle 0|_{B2} + p\,|e\rangle\langle e|_{B1} \otimes |1\rangle\langle 1|_{B2}$$

$$= (1-p)\,\rho_{B1} \otimes |0\rangle\langle 0|_{B2} + p\,(\mathrm{tr}\,\rho)\,|e\rangle\langle e|_{B1} \otimes |1\rangle\langle 1|_{B2}$$

drop '

this is called a "flagged" channel --
the output includes a classical system (B2 here)
labelling what channel has occurred to the input

Example: erasure channel (on a qubit)

$$\mathcal{E}_p(\rho) = (1-p)\,\rho_{B_1} \otimes |0\rangle\langle 0|_{B_2} + p\,(\mathrm{tr}\,\rho)\,|e\rangle\langle e|_{B_1} \otimes |1\rangle\langle 1|_{B_2}$$

To evaluate the 1-shot coherent info: take any $|\psi\rangle_{RA}$,

$$[I\otimes\mathcal{E}_p(|\psi\rangle\langle\psi|)]_{RB_1B_2} = (1-p)\,|\psi\rangle\langle\psi|_{RB_1} \otimes |0\rangle\langle 0|_{B_2} + p\,(\mathrm{tr}_B\,|\psi\rangle\langle\psi|)_R \otimes |e\rangle\langle e|_{B_1} \otimes |1\rangle\langle 1|_{B_2}$$

Recall when Bob has a classical system (B2 here), the coh info is a weighted average over this classical rand var (topic-5-1):

$$\therefore I_c(R\rangle B_1B_2)_{[I\otimes\mathcal{E}_p(|\psi\rangle\langle\psi|)]_{RB_1B_2}} = (1-p)\,I_c(R\rangle B_1)_{|\psi\rangle\langle\psi|_{RB_1}} + p\,I_c(R\rangle B_1)_{(\mathrm{tr}_B|\psi\rangle\langle\psi|)_R\otimes|e\rangle\langle e|_{B_1}}$$

S(B1) - S(RB1)

$$= (1-p)\,S\big((\mathrm{tr}_R|\psi\rangle\langle\psi|)_{B_1}\big) + p\,(-)\,S\big((\mathrm{tr}_B|\psi\rangle\langle\psi|)_R\big) = (1-2p)\,s$$

equal entropies, say, s

- $p \le \frac{1}{2}$, optimal $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $s = 1$
- $p > \frac{1}{2}$, $|\psi\rangle = |00\rangle$, $s = 0$

$\therefore Q^{(1)}(\mathcal{E}_p)$ = max(1-2p,0).

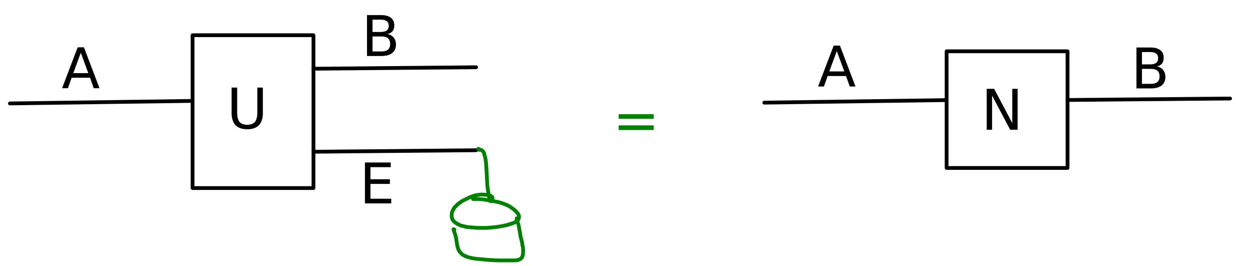
What about r-shot coherent info?
Will see it's equal to 1-shot coh info!

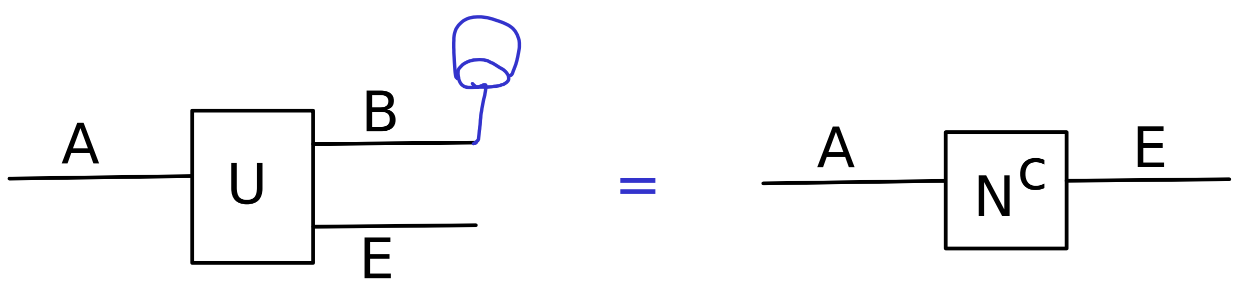
Complementary channel

Let N be any channel, U its isometric extension.

Def: A complementary channel of N, denoted N^c, is given by:

$$N^c(\rho) = \mathrm{tr}_B(U\rho U^\dagger)$$



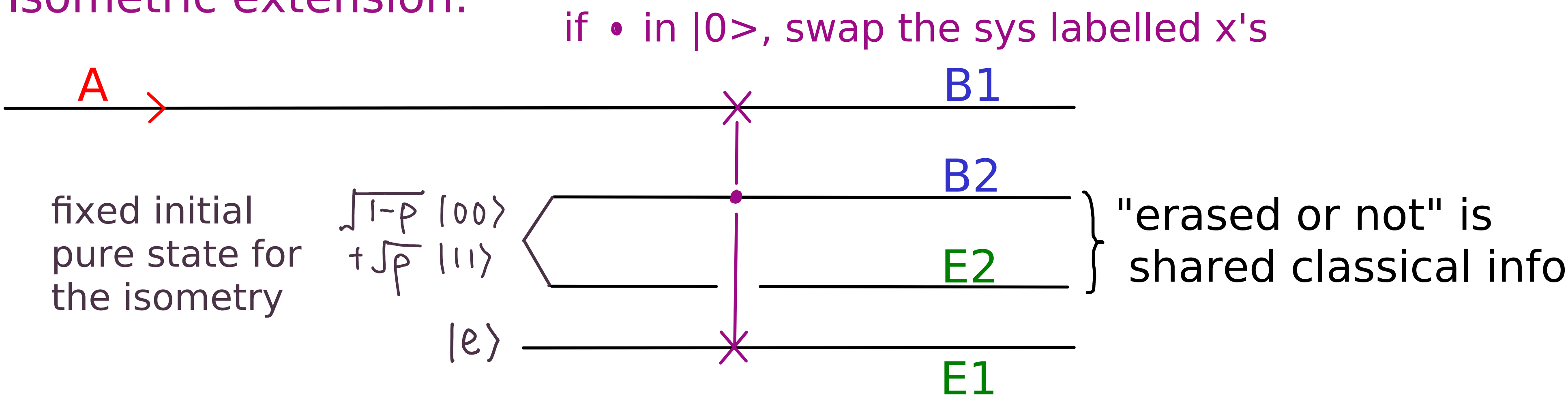


$(N^c)^c = N$ up to the un-important isometries

Example: erasure channel (on a qubit)

$$\mathcal{E}_p(\rho) = (1-p)\,\rho_{B1} \otimes |0\rangle\langle 0|_{B2} + p\,(\mathrm{tr}\,\rho)\,|e\rangle\langle e|_{B1} \otimes |1\rangle\langle 1|_{B2}$$

Isometric extension:



$$\mathcal{E}_p^c = \mathcal{E}_{1-p} \;!$$

Degradable channel

Let N be any channel, U its isometric extension.

Def: N is called degradable if $\exists \mathcal{D}$ (TCP map) s.t. $\mathcal{D} \circ N = N^c$.

Def: N is called anti-degradable if N^c is degradable.

Def: N is called symmetric if N is both degradable & antidegradable.

Intuition:

If N is degradable, "Bob is better than Eve"
(since Bob can post-process his channel output to obtain Eve's)

If N is antidegradable, "Eve is better than Bob"

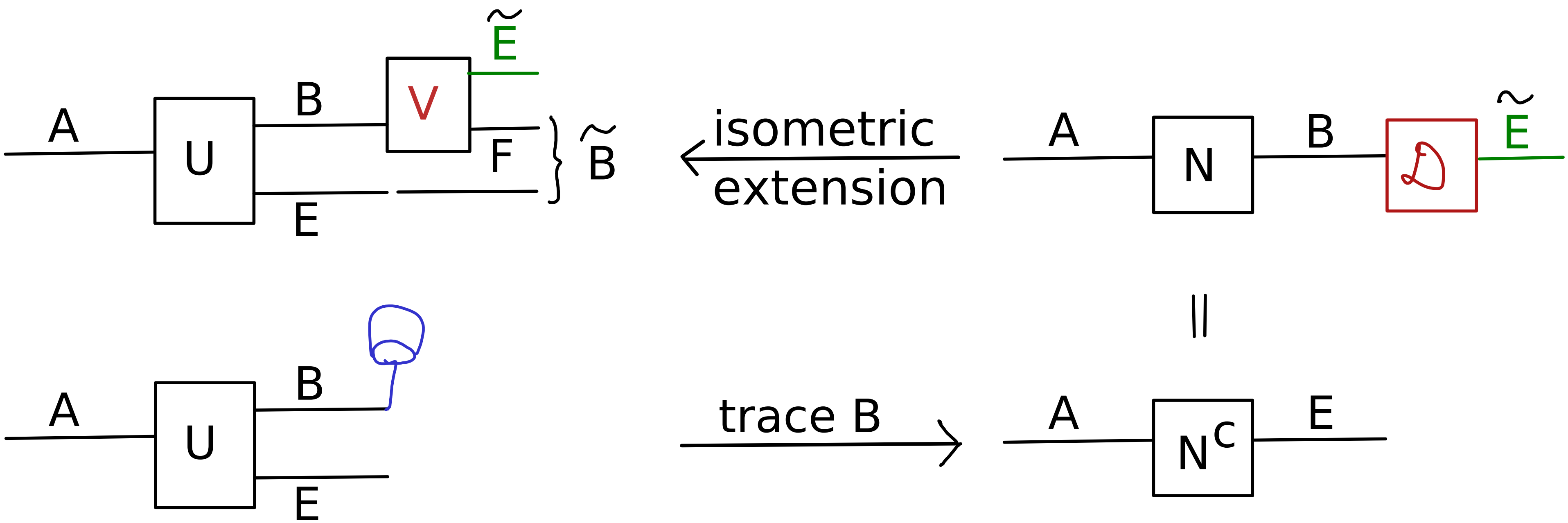
Degradable channel

Let N be any channel, U its isometric extension.

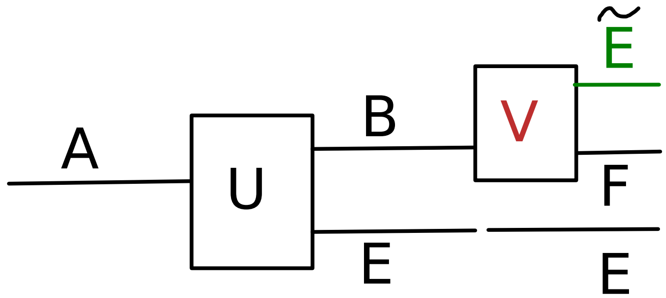
Def: N is called degradable if $\exists\, \mathcal{D}$ (TCP map) s.t. $\mathcal{D} \circ N = N^c$.

Def: $\mathcal{D}$ is some times called the degrading map.

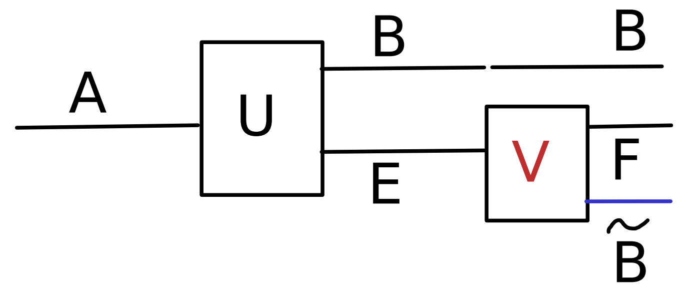
Let V be its isometric extension. When F is discarded, F goes to the env, thereby exchanging N and N^c



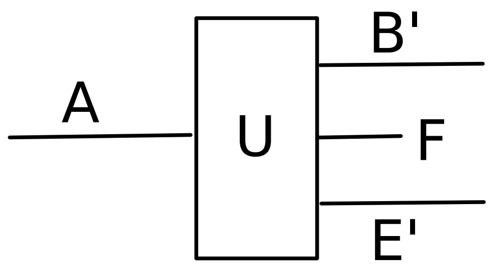
Degradable channel: $E\tilde{E}$ sym



Antidegradable channel: $B\tilde{B}$ sym



A characterization for degradable or antidegradable channels up to isometries of the output and env, is an isometric extension:



s.t., for all inputs on A (or equivalently for the Choi-state), the output is invariant under swapping B' E', and

for degradable channel: B = B'F, E = E'
antidegradable channel: B = B', E = E'F

Example: erasure channel (on a qubit)

$\mathcal{E}_p(\rho) = (1-p)\,\rho_{B1} \otimes |0\rangle\langle 0|_{B2} + p\,(\mathrm{tr}\,\rho)\,|e\rangle\langle e|_{B1} \otimes |1\rangle\langle 1|_{B2}$

To understand degradability of $\mathcal{E}_p$, try using $\tilde{\mathcal{E}}_q$ as a degrading map

where $\tilde{\mathcal{E}}_q$ 1. apply $\mathcal{E}_q$ to B1

2. replace the 2 erasure flags with their "or"

Claim: $\tilde{\mathcal{E}}_q \circ \mathcal{E}_p = \mathcal{E}_{p+q-pq}$

Proof: $\tilde{\mathcal{E}}_q \circ \mathcal{E}_p$ is an erasure channel

no erasure with prob (1-p)(1-q) = 1-p-q+pq

so, prob of erasure = p+q-pq

Recall $\mathcal{E}_p^c = \mathcal{E}_{1-p}$ which equals to $\tilde{\mathcal{E}}_q \circ \mathcal{E}_p = \mathcal{E}_{p+q-pq}$

if 1-p = p+q-pq or (1-2p) = (1-p) q

if $p \leq \frac{1}{2}$, $\tilde{\mathcal{E}}_{q=\frac{1-2p}{1-p}}$ is a degrading map for $\mathcal{E}_p$. $\therefore$ $\mathcal{E}_p$ degradabe for $p \leq \frac{1}{2}$

if $p \geq \frac{1}{2}$, $\mathcal{E}_p^c = \mathcal{E}_{1-p}$ degradable. $\therefore$ $\mathcal{E}_p$ antidegradable.

$\mathcal{E}_{\frac{1}{2}}$ is symmetric (also called the 50-50 erasure channel)

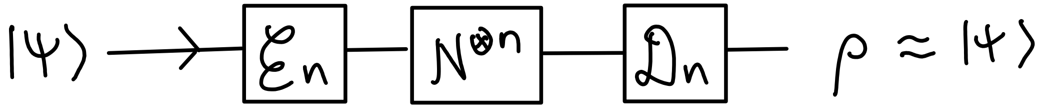
Theorem If N is antidegradable, then Q(N) = 0.

Theorem' If N is antidegradable, then one cannot send a single qubit with arbitrarily large number of uses of N.

Intuition: if there is a coding scheme transmitting quantum data to Bob, Eve can decode a copy too, implying cloning.

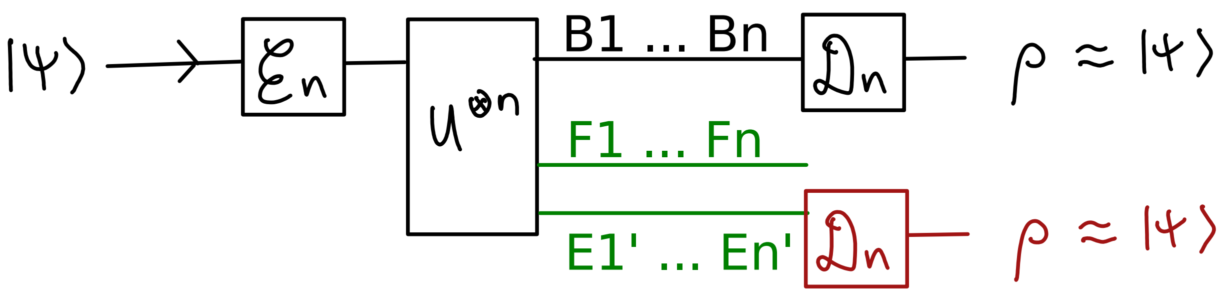
Proof (theorem'), by contradiction

Suppose there is some n, and a coding scheme that transmits one qubit with n uses of N with very small error.



expanding N into its isometric extension:

"black" -- by hypothesis



by symmetry of B1 ... Bn, E1' ... En', applying $\mathcal{D}_n$ to E1' ... En' gives ρ

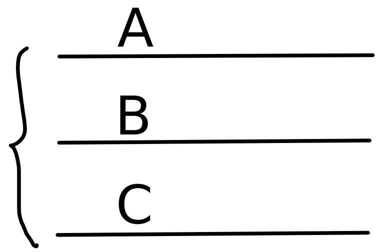
Joint state on B1 ... Bn E1' ... En' $\approx |\psi\rangle^{\otimes 2}$, contradicting no-cloning thm.

Remark on the last argument:

On 2 sys, if each has reduced state ρ, joint state need not be $\rho^{\otimes 2}$.

e.g.,

$\frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$



$\leftarrow$ reduced state $\frac{I}{2}$ (A)

$\leftarrow$ reduced state $\frac{I}{2}$ (C)

Joint state on AC: $\frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) \neq \frac{I}{2} \otimes \frac{I}{2}$.

Remark on the last argument:

In our problem, we use the fact ρ is close to a pure state to conclude.

A ← reduced state $\rho \approx |\psi\rangle\langle\psi|$ (1)

B ← purifying system

C ← reduced state $\rho \approx |\psi\rangle\langle\psi|$ (2)

From (1), Uhlmann's thm, relation between purifications, and the fact joint state on ABC $|\Psi\rangle$ and $|\psi\rangle_A \otimes |0\rangle_{BC}$ both approx purifies A

$\exists W$ unitary s.t. $|\Psi\rangle \approx (I_A \otimes W_{BC})\,|\psi\rangle_A \otimes |0\rangle_{BC} = |\psi\rangle_A \otimes |\alpha\rangle_{BC}$

From (2), Uhlmann's thm, and relation between purifications, $|\alpha\rangle_{BC}$ and $|0\rangle_B |\psi\rangle_C$ both approx purifies C

$\exists U$ unitary s.t. $|\alpha\rangle_{BC} \approx (U_B \otimes I_C)\,|0\rangle_B |\psi\rangle_C = |\beta\rangle_B \otimes |\psi\rangle_C$

$\therefore$ joint state on ABC $|\Psi\rangle \approx |\psi\rangle_A \otimes |\beta\rangle_B \otimes |\psi\rangle_C$

$\therefore$ joint state on AC $\approx |\psi\rangle_A \otimes |\psi\rangle_C$ or $\rho^{\otimes 2}$

Theorem If N is antidegradable, then Q(N) = 0.

Theorem' If N is antidegradable, then one cannot send a single qubit with arbitrarily large number of uses of N.

Corollary 1 $Q(\mathcal{E}_p) = 0 \quad \forall p \geq \frac{1}{2}$

Recall noiseless classical channel: $|0\rangle_A \to |00\rangle_{BE}$ so it's symmetric.

$|1\rangle_A \to |11\rangle_{BE}$

Corollary 2: classical channels have 0 quantum capacity.

In fact, cannot comm 1 qubit even with arbitrarily many uses.

Theorem If N1 and N2 are degradable,

then $Q^{(1)}(N_1 \otimes N_2) = Q^{(1)}(N_1) + Q^{(2)}(N_2)$

Corollary If N is degradable, then Q(N) = $Q^{(1)}(N)$!

$\because \forall r \quad Q^{(r)}(N) = Q^{(1)}(N)$

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The proof relies on the following two lemmas.

Lemma 1: for any state on 4 systems RTXY

(i) $S(RT|XY) \leq S(R|X) + S(T|Y)$

(ii) with equality if the state is a product across RX / TY .

Proof: RHS - LHS

$= S(R|X) + S(T|Y) - S(RT|XY)$

$= S(RX) - R(X) + S(TY) - S(Y) - [S(RTXY) - S(XY)]$

$= S(RX:TY) - S(X:Y) \geq 0$ so (i) holds

tr R tr T QMI nonincreasing under tracing

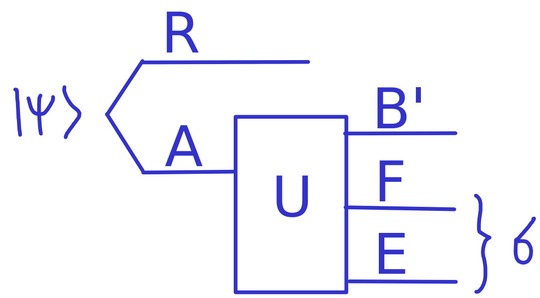
$S(X:TY)$

If state is a product across RX / TY, $S(RX:TY) = 0$, $S(X:Y) = 0$ so equality holds, proving (ii).

Lemma 2: If N is degradable, $|\psi\rangle_{RA}$ is any input,

then $I_c(R\rangle B)_{I\otimes N(|\psi\rangle\langle\psi|)} = S(F|E)_\sigma$

where



output space = B = B'F,
B' and E are symmetric

Proof: $I_c(R\rangle B)_{I\otimes N(|\psi\rangle\langle\psi|)}$

$$= S(B) - S(E) = S(B'F) - S(E) = S(EF) - S(E) = S(F|E)_\sigma$$

degradability, B' & E symmetric

Recall in general $I_c(R\rangle B) = S(B) - S(RB) = - S(R|B)$

So, for degradable channel, the coherent info exhibits properties "opposite" to usual (e.g., subadditive not superadditive ... as we'll see)

Theorem If N1 and N2 are degradable,

then $Q^{(1)}(N_1 \otimes N_2) = Q^{(1)}(N_1) + Q^{(2)}(N_2)$

Proof: $[\geq]$ Let $|\psi_1\rangle_{R_1A_1}$ attain the max of $I_c(R_1 \rangle B_1)_{I \otimes N_1(|\psi\rangle\langle\psi|)}$

Similarly for $|\psi_2\rangle_{R_2A_2}$.

$$Q^{(1)}(N_1 \otimes N_2) \geq I_c(R_1R_2 \rangle B_1B_2)_{\underbrace{I_{R_1R_2} \otimes N_1 \otimes N_2 (|\psi_1\rangle\langle\psi_1|_{R_1A_1} \otimes |\psi_2\rangle\langle\psi_2|_{R_2A_2})}_{\text{product state over R1B1 / R2B2}}}$$

$\|$

-S(R1R2|B1B2)

$\|$ lemma 1 (ii)

-S(R1|B1) - S(R2|B2)

$\|$

$$I_c(R_1 \rangle B_1)_{I_{R_1} \otimes N_1(|\psi_1\rangle\langle\psi_1|_{R_1A_1})} + I_c(R_2 \rangle B_2)_{I_{R_2} \otimes N_2(|\psi_2\rangle\langle\psi_2|_{R_2A_2})}$$

$\|$ optimality of $|\psi_1\rangle, |\psi_2\rangle$

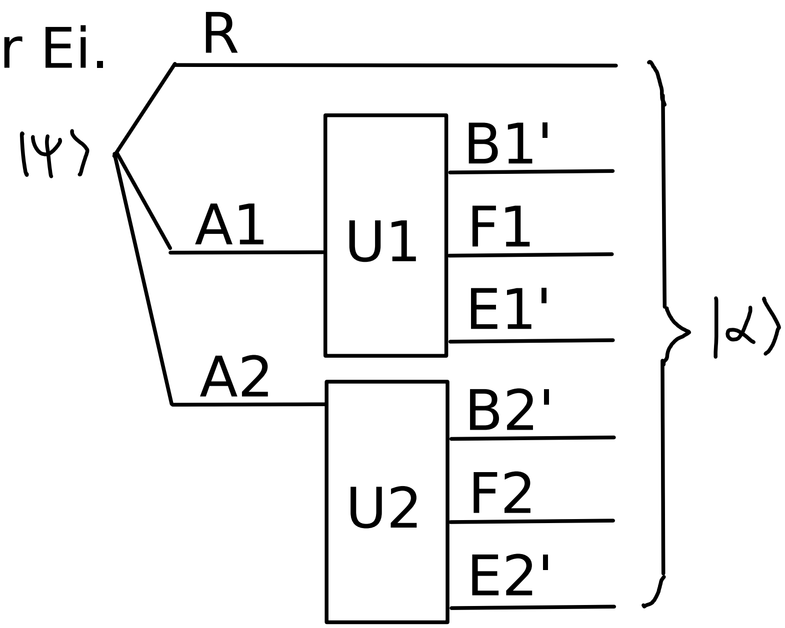
$$Q^{(1)}(N_1) + Q^{(2)}(N_2)$$

Theorem If N1 and N2 are degradable,

then $Q^{(1)}(N_1 \otimes N_2) = Q^{(1)}(N_1) + Q^{(2)}(N_2)$

Proof: [$\leq$] Let $|\psi\rangle_{RA_1A_2}$ be the optimal input for $N_1 \otimes N_2$

Let Bi = Bi' Fi, Ei = Ei' be output & env for Ei.



$Q^{(1)}(N_1 \otimes N_2)$

$= I_c(R \rangle B_1 B_2)_{|\alpha\rangle}$

= S(F1F2 | E1E2) by lemma 2

$\leq$ S(F1 | E1) + S(F2 | E2) by lemma 1 (i)

Theorem If N1 and N2 are degradable,

then $Q^{(1)}(N_1 \otimes N_2) = Q^{(1)}(N_1) + Q^{(2)}(N_2)$

Proof: [$\leq$] Let $|\psi\rangle_{RA_1A_2}$ be the optimal input for $N_1 \otimes N_2$

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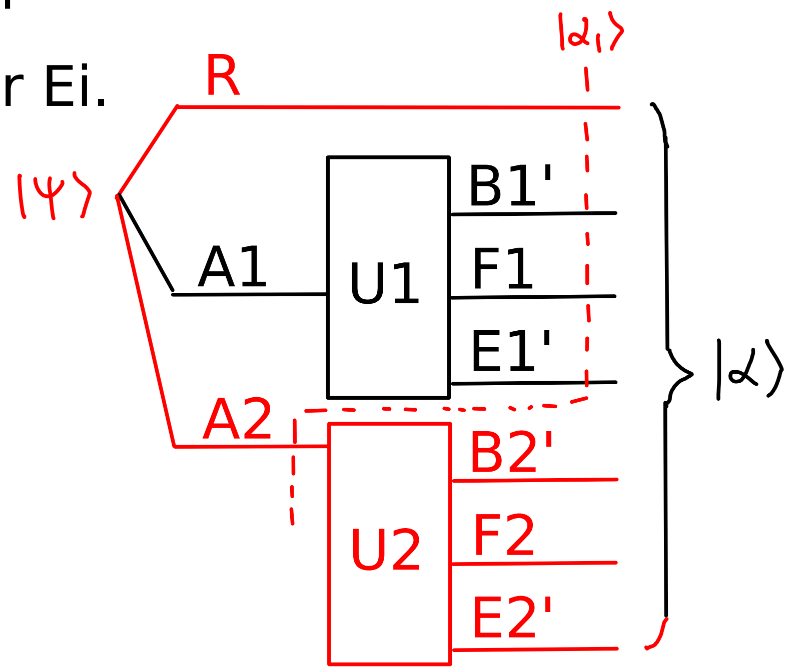


$Q^{(1)}(N_1 \otimes N_2)$

$= I_c(R > B_1 B_2)_{|\alpha\rangle}$

= S(F1F2 | E1E2) by lemma 2

$\leq$ S(F1 | E1) + S(F2 | E2) by lemma 1 (i)

$= I_c(R_1 > B_1)_{|\alpha_1\rangle} + I_c(RA_1 > B_2)_{|\alpha_2\rangle}$

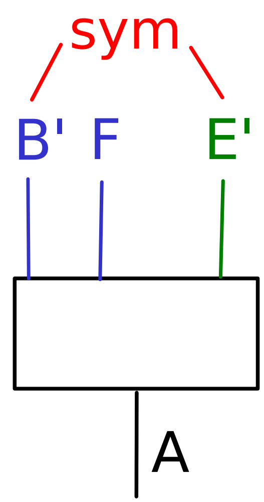
$R_1 = RA_2$

$\leq Q^{(1)}(N_1) + Q^{(2)}(N_2)$

Summary: N degradable $\Longleftrightarrow$ N^c antidegradable

$Q(N) = Q^{(1)}(N)$ $\qquad\qquad$ $Q(N^c) = 0$

output for N



output for N^c

sum<1 for p>0

e.g., erasure channel

$\mathcal{E}_p$, $p \le \frac{1}{2}$, $Q(\mathcal{E}_p) =$ 1-2p $\qquad$ $\mathcal{E}_{1-p}$, $p \le \frac{1}{2}$, $Q(\mathcal{E}_{1-p}) = 0$

e.g., dephasing channel $\quad Z_p(\rho) = (1-p)\rho + p Z \rho Z$

Z_p, $p \in [0,1]$, $Q(Z_p) =$ 1-h(p) $\qquad$ $Q(Z_p^c) = 0$

(see 2016 lecture 18)

e.g., amplitude damping channel (see A4)