CO781 / QIC 890:

Theory of Quantum Communication

Topic 5, part 3

Transmitting quantum data through a quantum channels

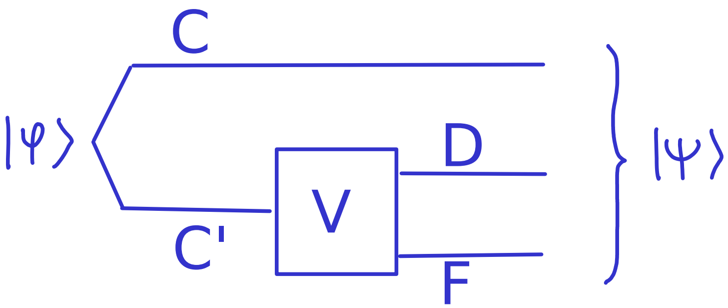
- the proof of the LSD theorem outline
- the decoupling approach (exact)
- the decoupling approach (approx)
- the decoupling condition (1-shot)

Tue

- the direct coding theorem for the LSD theorem
- typicality for the direct coding theorem
- the decoupling condition (applied to direct coding theorem)
- the converse

Last time

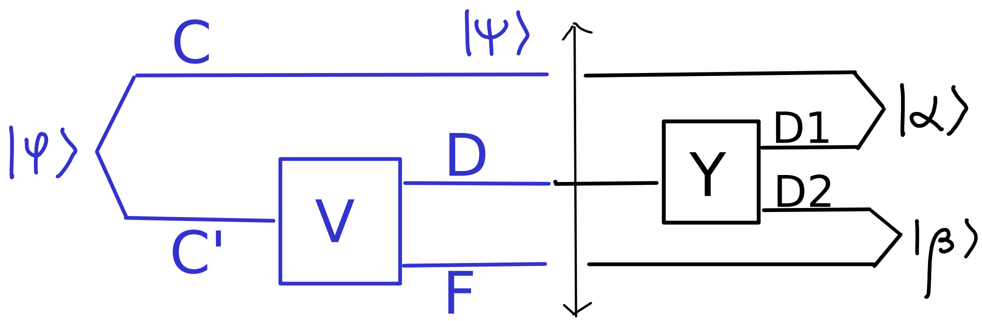
Decoupling approach (approximate case via Uhlmann's thm)



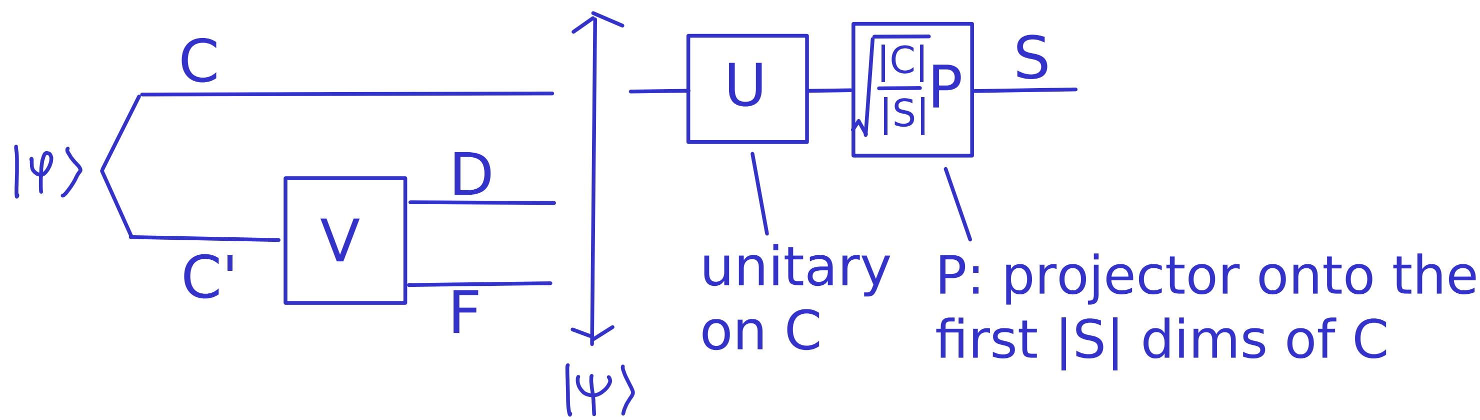
Lemma: if $\| \psi^{CF} - \alpha^{C} \otimes \beta^{F} \|_1 \le \epsilon$

then $\exists$ isometry $Y: D \to D1\, D2$,

$F(I_{CF} \otimes Y_D |\psi\rangle , |\alpha\rangle_{CD_1} \otimes |\beta\rangle_{D_2 F}) \ge 1 - \frac{\epsilon}{2}$



Decoupling condition (1-shot):



$\Big\}\ |\xi\rangle_{SDF}$

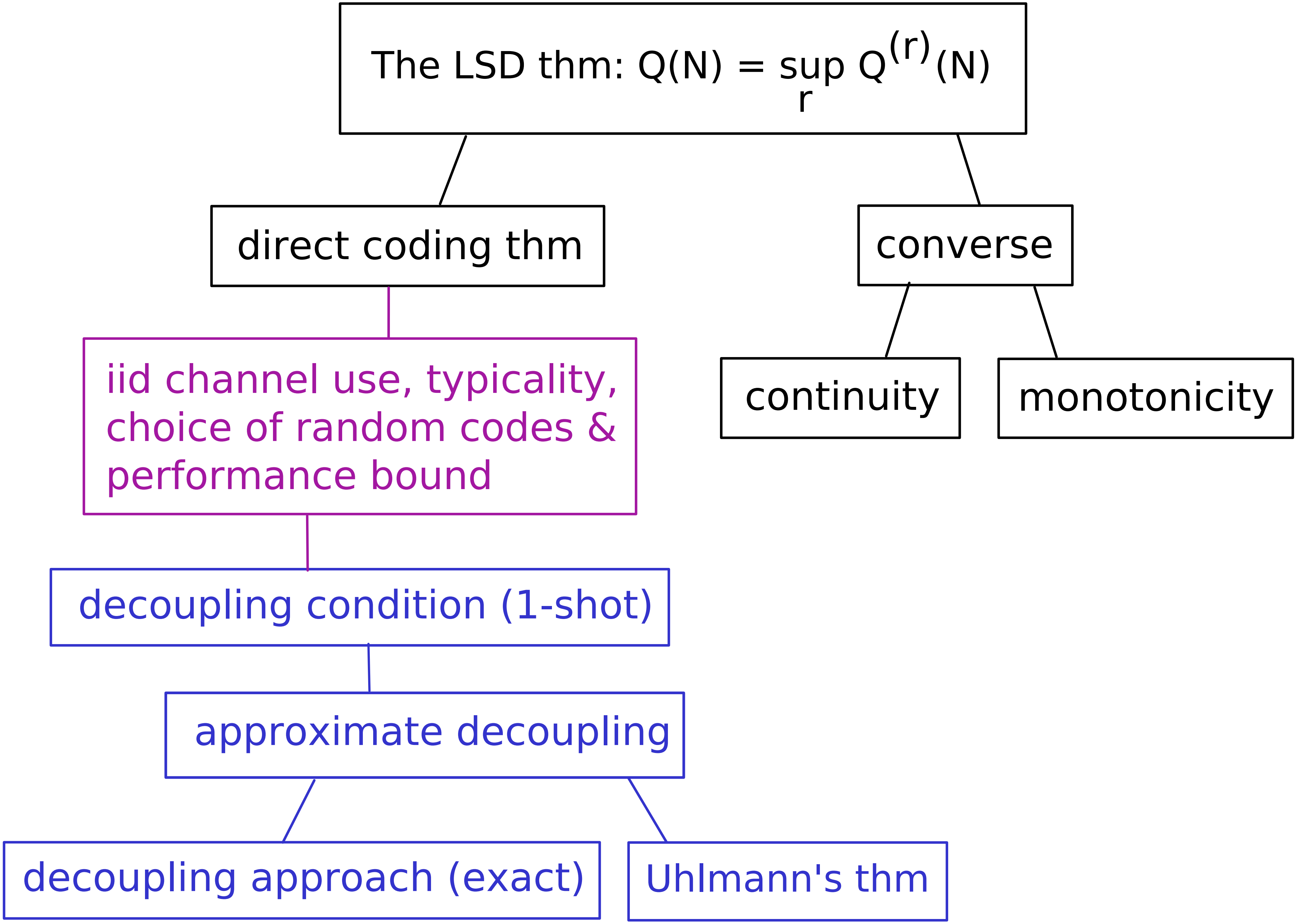
Let $|\xi\rangle = \sqrt{\frac{|C|}{|S|}}\, P U |\psi\rangle$ (a vector not normalized, on SDF)

Theorem: if U chosen according to the Haar measure,

then $\displaystyle \mathop{\mathbb{E}}_{U} \left\| \xi^{SF} - \pi^{S} \otimes \psi^{F} \right\|_1 \leq |S|^{\frac{1}{2}}\, |F|^{\frac{1}{2}} \left(\mathrm{Tr}\left[(\psi^{D})^2 \right] \right)^{\frac{1}{2}}$

not normalized

max mixed state on S

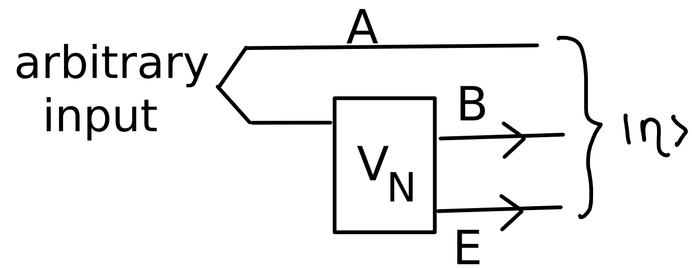

The LSD thm: $Q(N) = \sup_r Q^{(r)}(N)$
direct coding thm
converse
iid channel use, typicality, choice of random codes & performance bound
continuity
monotonicity
decoupling condition (1-shot)
approximate decoupling
decoupling approach (exact)
Uhlmann's thm

Direct coding theorem: For any channel N, any input, the 1-shot coherent info is an achievable rate for entanglement generation

First define (M,n) codes :

V_N isometric extension of the channel

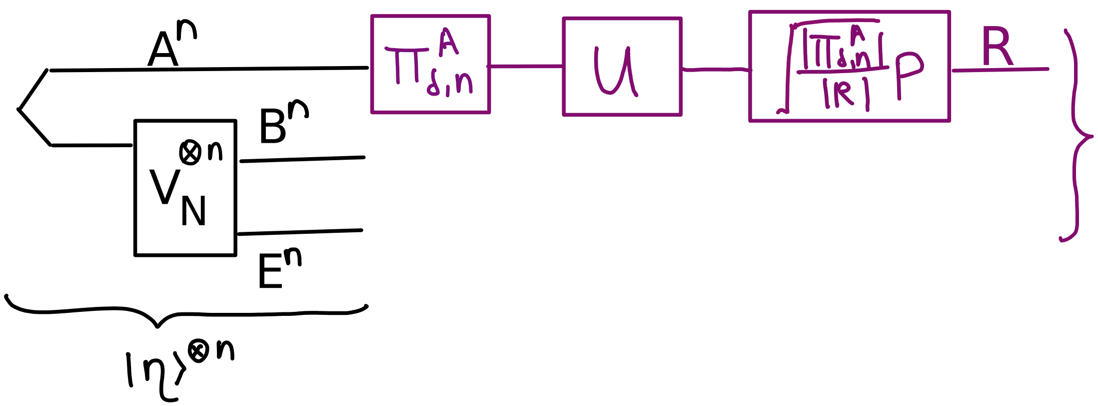
$|\eta\rangle_{ABE}$ output after N acts on arbitrary input 1-use



$$(|\eta\rangle_{ABE})^{\otimes n} = (|\eta\rangle^{\otimes n})_{A^n B^n E^n}$$

$\Pi^A_{\delta,n}$ = δ-weakly- projector onto typical space of $(\eta_A)^{\otimes n}$, $\Pi^B_{\delta,n}$, $\Pi^E_{\delta,n}$ similar.

$\mathrm{tr}_{BE}|\eta\rangle\langle\eta|$



$|\xi\rangle_{RB^nE^n}$ not normalized

$\Psi = \frac{|\xi\rangle\langle\xi|}{\langle\xi|\xi\rangle}$ normalized

Claim: $\mathbb{E}_U \left\| \Psi^{RE^n} - \Pi_R \otimes \eta^{E^n} \right\|_1 \leq \epsilon_n \downarrow 0$ for $|R| = 2^{n(I(A\rangle B)_\eta - 5\delta)}$

NB. If claim holds, can decode B^n to get log |R| ebits on R and some B1.

Proof of claim:

(1) $\| \Psi^{RE^n} - \pi_R \otimes \eta^{E^n} \|_1 \leq 2 \| \xi^{RE^n} - \pi_R \otimes \eta^{E^n} \|_1$

This is due to a general lemma:

For any density matrices ρ, σ, any c $\in \mathbb{R}$, $\|\rho - \sigma\|_1 \leq 2\|c\rho - \sigma\|_1$

Proof of lemma:

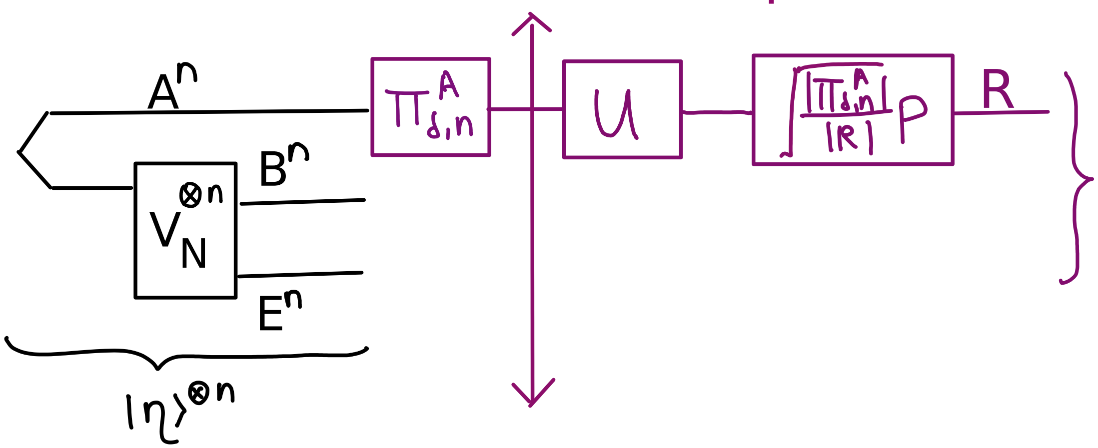
$$\|\rho - c\rho\|_1 = |1-c| = |\mathrm{tr}(c\rho - \sigma)| \leq \|c\rho - \sigma\|_1$$

$$\|\rho - \sigma\|_1 \leq \|\rho - c\rho\|_1 + \|c\rho - \sigma\|_1 \leq 2\|c\rho - \sigma\|_1$$

Proof of claim:

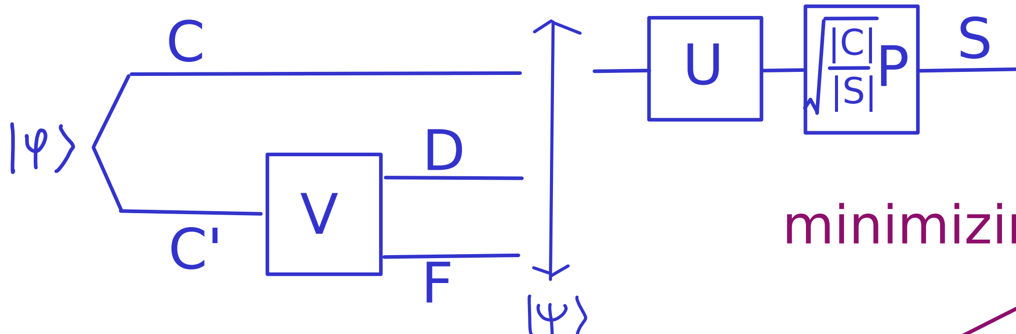
(1) $\left\| \Psi^{RE^n} - \pi_R \otimes \eta^{E^n} \right\|_1 \leq 2 \left\| \xi^{RE^n} - \pi_R \otimes \eta^{E^n} \right\|_1$

(2) the state to be shown decoupled



$|\xi\rangle_{RB^nE^n}$ not normalized

decoupling condition from Tue



so we want to trim B^n, E^n

minimizing these gives a bigger |S|

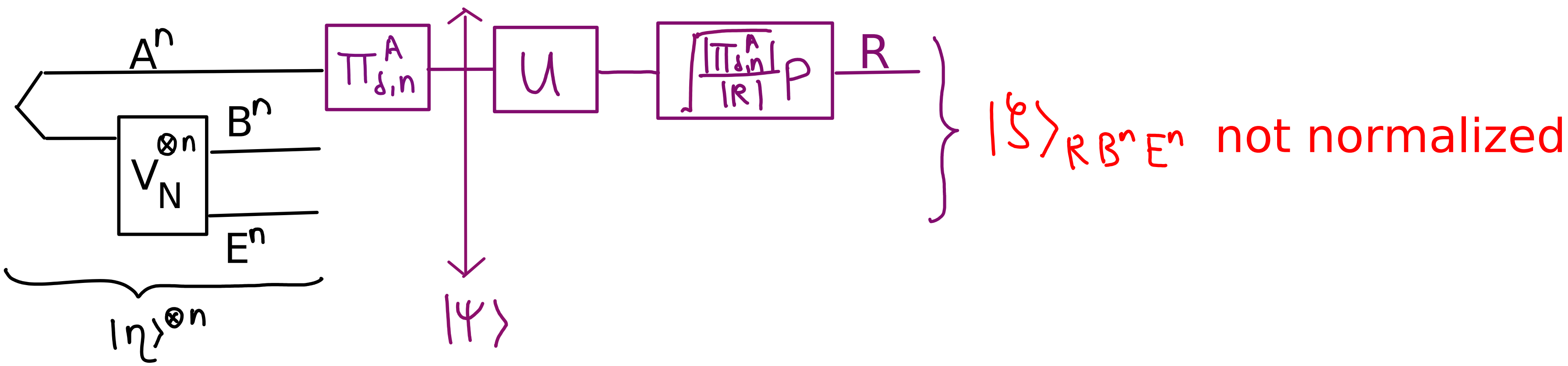
while keeping RHS small

$$\mathbb{E}_U \left\| \xi^{SF} - \pi^S \otimes \psi^F \right\|_1 \leq |S|^{\frac{1}{2}} |F|^{\frac{1}{2}} \left(\mathrm{Tr}\left[(\psi^D)^2 \right] \right)^{\frac{1}{2}}$$

Proof of claim:

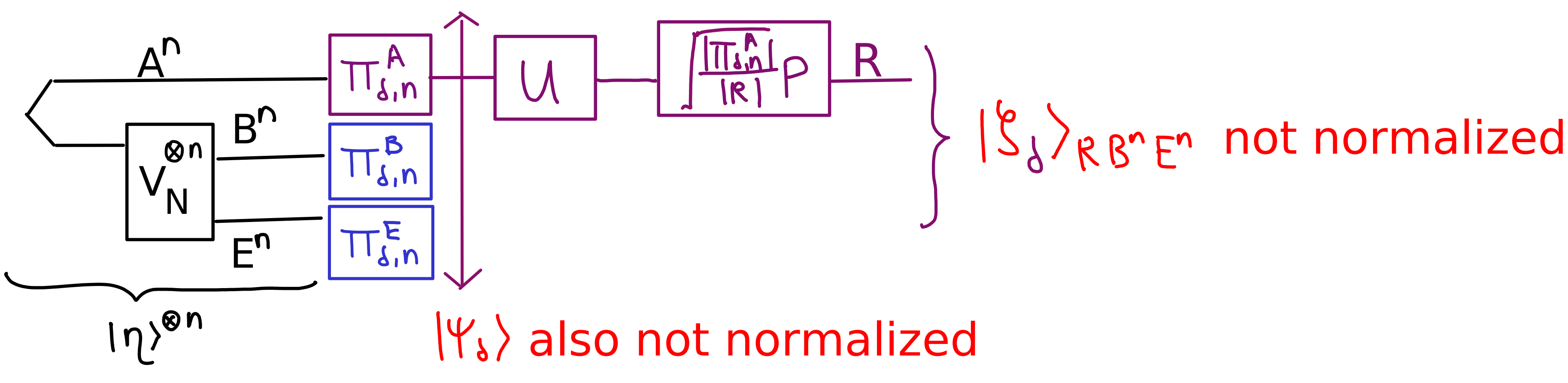
(1) $\left\| \Psi^{RE^n} - \pi_R \otimes \eta^{E^n} \right\|_1 \leq 2 \left\| \xi^{RE^n} - \pi_R \otimes \eta^{E^n} \right\|_1$

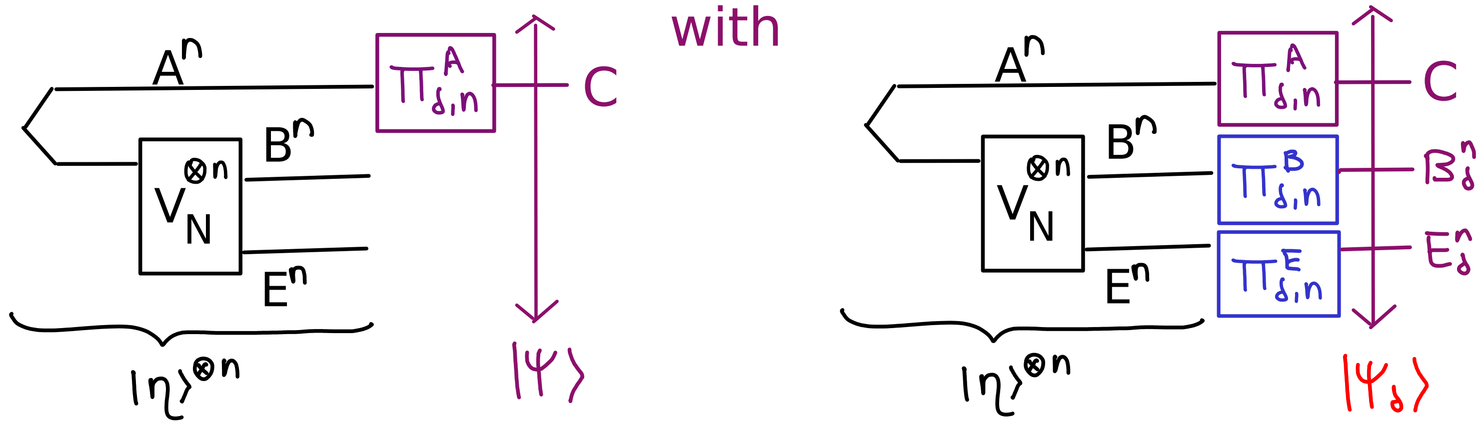
(2) the state to be shown decoupled



in a fantansy word (for proving things):

trimming B^n E^n to suppress RHS in the decoupling condition



Compare  with

Recall the analysis of the TTS (transmit typical space) protocol for entanglement dilution in topic-2-3.pdf (p3-5):

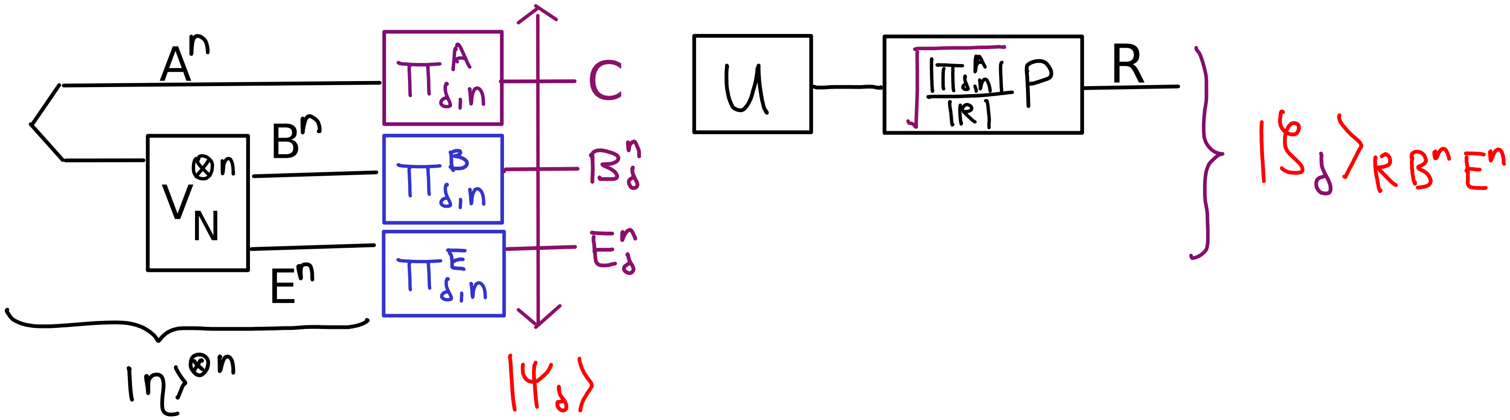
$$\left\| \left(|\psi\rangle\langle\psi|\right)^{\otimes n} - I\otimes\Pi_s \left(|\psi\rangle\langle\psi|\right)^{\otimes n} I\otimes\Pi_s \right\|_1$$

$$\leq 2\sqrt{1-\left(\langle\psi|^{\otimes n}\,(I\otimes\Pi_s)\,|\psi\rangle^{\otimes n}\right)^2} \leq 2\sqrt{2}\sqrt{\epsilon}$$

$\forall\ \delta>0,\ \epsilon>0$ for large n, $|\eta\rangle^{\otimes n}$, $|\psi\rangle$, $|\psi_\delta\rangle$ all "ϵ" close to one another (in trace norm).

(T1)

For $|\psi_\delta\rangle$,



$$|E^n_\delta| \leq 2^{n(S(E)_\eta+\delta)}, \quad |B^n_\delta| \leq 2^{n(S(B)_\eta+\delta)}$$

on B^n_δ eigenvalues $\in \left[2^{-n(S(B)_\eta+\delta)},\ 2^{-n(S(B)_\eta-\delta)}\right]$

$$\mathrm{tr}\left(\psi_\delta^{B^n}\right)^2 \leq |B^n_\delta| \cdot \|\psi_\delta^{B^n}\|_\infty^2 \leftarrow \text{max eval}$$

$$\leq 2^{n(S(B)_\eta+\delta)}\, 2^{-n(S(B)_\eta-\delta)\cdot 2} = 2^{-n(S(B)_\eta+3\delta)}$$

Apply $\frac{|\Pi^A_{\delta,n}|}{|R|} P\mathcal{U}$ to C & use decoupling condition $(S\to R,\ F\to E^n_\delta,\ D\to B^n_\delta)$

$$\text{(T2)} \quad \mathop{\mathbb{E}}_{\mathcal{U}} \left\| \xi_\delta^{RE^n_\delta} - \pi^R \otimes \psi_\delta^{E^n_\delta} \right\|_1 \leq |R|^{\frac{1}{2}}\, |E^n_\delta|^{\frac{1}{2}} \left(\mathrm{Tr}\left[\left(\psi_\delta^{B^n_\delta}\right)^2\right]\right)^{\frac{1}{2}}$$

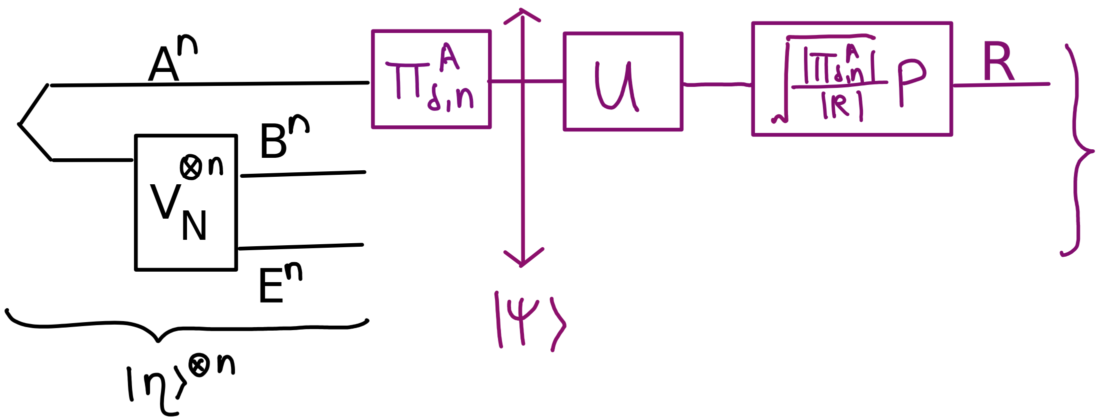
$$\leq \left[2^{n(I(A\rangle B)_\eta-5\delta)}\, 2^{n(S(E)_\eta+\delta)}\, 2^{-n(S(B)_\eta+3\delta)}\right]^{\frac{1}{2}} = 2^{-3n\delta/2}$$

So things are good in the fantasy world with $|\psi_\delta\rangle$ ... now back to $|\psi\rangle$.

Proof of claim:

(1) $\left\| \Psi^{RE^n} - \Pi_R \otimes \eta^{E^n} \right\|_1 \leq 2 \left\| \xi^{RE^n} - \Pi_R \otimes \eta^{E^n} \right\|_1$

(2)



$|\xi\rangle_{RB^nE^n}$ not normalized

$$\left\| \xi^{RE^n} - \Pi_R \otimes \eta^{E^n} \right\|_1$$

$$\leq \underbrace{\left\| \xi^{RE^n} - \xi_\delta^{RE^n} \right\|_1}_{\text{cannot just bound by (T1) because projector not trace preserving}} + \underbrace{\left\| \xi_\delta^{RE_\delta^n} - \Pi_R \otimes \psi_\delta^{E_\delta^n} \right\|_1}_{\mathbb{E}_U \ldots \leq 2^{-3n\delta/2} \text{ by (T2)}} + \underbrace{\left\| \Pi_R \otimes \psi_\delta^{E^n} - \Pi_R \otimes \eta^{E^n} \right\|_1}_{= \left\| \psi_\delta^{E^n} - \eta^{E^n} \right\|_1 \leq \epsilon \text{ by (T1)}}$$

cannot just bound by (T1) because projector not trace preserving

$\mathbb{E}_U \,\ldots \leq 2^{-3n\delta/2}$ by (T2)

$= \left\| \psi_\delta^{E^n} - \eta^{E^n} \right\|_1 \leq \epsilon$ by (T1)

Lemma: W random operator on finite Hilbert space, $\mathbb{E}\, W^\dagger W \leq I$, X hermitian.

Then, $\mathbb{E}\, \| W X W^\dagger \|_1 \leq \| X \|_1$.

Pf: (1) useful fact $\| M \|_1 = \max \{ \operatorname{tr} MY ,\ -I \leq Y \leq I \}$

(2) if $-I \leq Y \leq I$

then $-I \leq -\mathbb{E}\, W^\dagger W \leq \mathbb{E}\, W^\dagger Y W \leq \mathbb{E}\, W^\dagger W \leq I$

(3) for each W, from (1)

$\exists\ -I \leq Y_W \leq I$ s.t. $\| W X W^\dagger \|_1 = \operatorname{tr} Y_W W X W^\dagger = \operatorname{tr} X W^\dagger Y_W W$

$$\mathbb{E}\, \| W X W^\dagger \|_1 = \operatorname{tr} X \underbrace{\mathbb{E}\, W^\dagger Y_W W}_{\text{in } [-I, I]} \underset{\uparrow\ (1)\text{ again}}{\leq} \| X \|_1$$

Lemma: W random operator on finite Hilbert space, $\mathbb{E}\, W^\dagger W \leq I$, X hermitian.

Then, $\mathbb{E}\, \| W X W^\dagger \|_1 \leq \| X \|_1$.

Apply lemma to $\| \varrho^{REn} - \varrho_\delta^{REn} \|_1$

Choose $W = \sqrt{\frac{|\Pi^{A}_{\delta, n}|}{|R|}}\, P\, U$

completely randomizing map

$$\mathbb{E}_U\, W^\dagger W = \frac{|\Pi^{A}_{\delta, n}|}{|R|}\, \mathbb{E}_U\, U^\dagger P U = \frac{|\Pi^{A}_{\delta, n}|}{|R|} (\operatorname{tr} P) \frac{I_C}{|C|} = I_C$$

$$X = \psi^{REn} - \psi_\delta^{REn}$$

$$W X W^\dagger = \varrho^{REn} - \varrho_\delta^{REn}$$

So from lemma: $\mathbb{E}\, \| W X W^\dagger \|_1 \leq \| X \|_1$

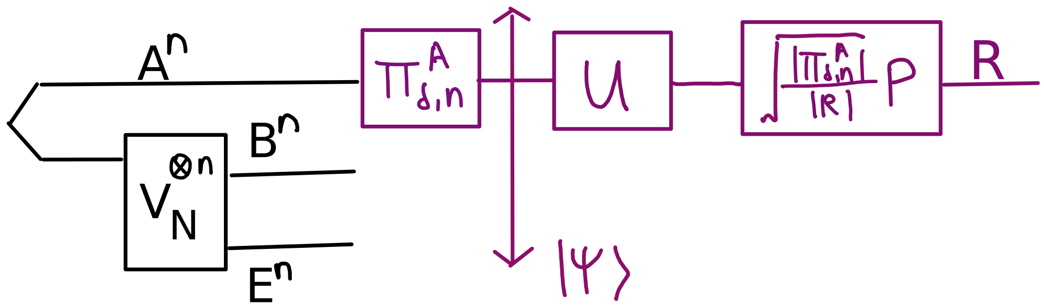
(T1) finally applies

$$\mathbb{E}_U\, \| \varrho^{REn} - \varrho_\delta^{REn} \|_1 \qquad \| \psi^{REn} - \psi_\delta^{REn} \|_1 \leq \epsilon$$

Proof of claim:

(1) $\| \Psi^{RE^n} - \pi_R \otimes \eta^{E^n} \|_1 \le 2 \, \| \xi^{RE^n} - \pi_R \otimes \eta^{E^n} \|_1$

(2)



$|\xi\rangle_{RB^nE^n}$ not normalized

$$\| \xi^{RE^n} - \pi_R \otimes \eta^{E^n} \|_1$$

$$\le \underbrace{\| \xi^{RE^n} - \xi_\delta^{RE^n} \|_1}_{\mathbb{E}_U \cdots \le \epsilon} + \underbrace{\| \xi_\delta^{RE_\delta^n} - \pi_R \otimes \psi_\delta^{E_\delta^n} \|_1}_{\mathbb{E}_U \cdots \le 2^{-3n\delta/2}} + \underbrace{\| \pi_R \otimes \psi_\delta^{E^n} - \pi_R \otimes \eta^{E^n} \|_1}_{= \| \psi_\delta^{E^n} - \eta^{E^n} \|_1 \le \epsilon}$$

previous page + (T1) by (T2) by (T1)

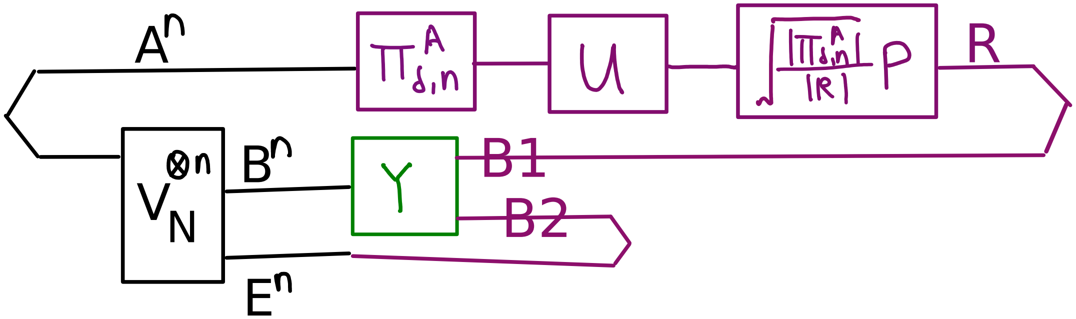
(3) together, $\underbrace{\mathbb{E}_U \| \Psi^{RE^n} - \pi_R \otimes \eta^{E^n} \|_1}_{} \le 2 \, \mathbb{E}_U \| \xi^{RE^n} - \pi_R \otimes \eta^{E^n} \|_1$

$\le 4\epsilon + 2 \cdot 2^{-3n\delta/2} \downarrow 0$

(4) $\downarrow$ 0 for some U

Using approximate decoupling approach: $\exists Y : B^n \to B_1 B_2$

Approx:

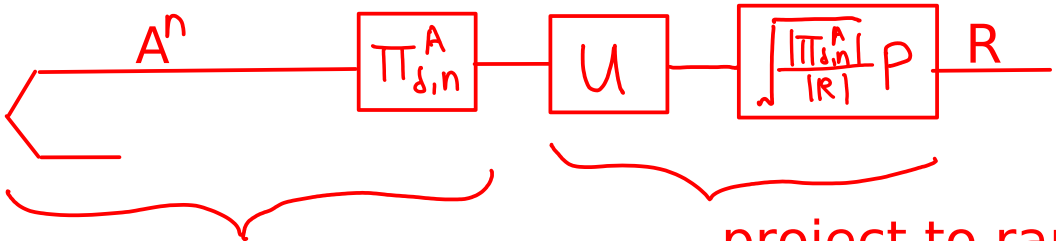


Also, reduced state on R is closed to π^R

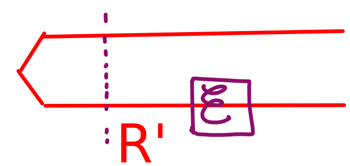
so state on R B1 is closed to max entangled state

Actual code:

R = random subspace of C w/ max mixed reduced state



$\approx$



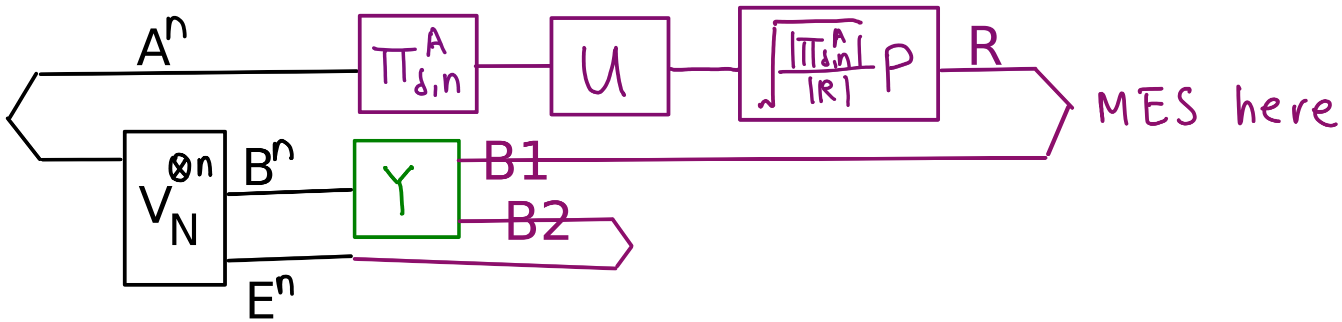
close to max entangled state on two copies of the typical space C for $\rho_A^{\otimes n}$

project to random subspace of C

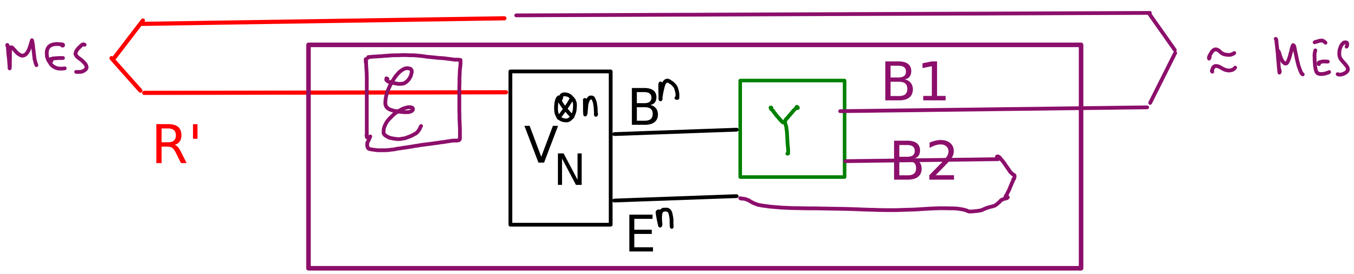
standard max ent state (over typical spaces RR') + unitary encoding on R

Using approximate decoupling approach: $\exists\, Y : B^n \to B_1 B_2$

Approx:



R = random subspace of C



So, the TCP map from R' to B1 has Choi-state close to MES.
R' can be trimmed to a smaller good code for transmitting quantum data.

Direct coding theorem: For any channel N, any input, the 1-shot coherent info is an achievable rate for entanglement generation

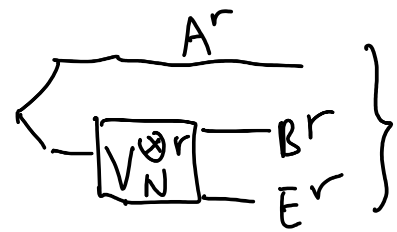
and for transmitting quantum data ...

Now optimize over 1-use input gives a code that achieves the 1-shot coherent info for the channel N.

The r-shot coherent information is also achievable:

Code for $\mathcal{N}^{\otimes r}$

pick any r-use input, take n copies of



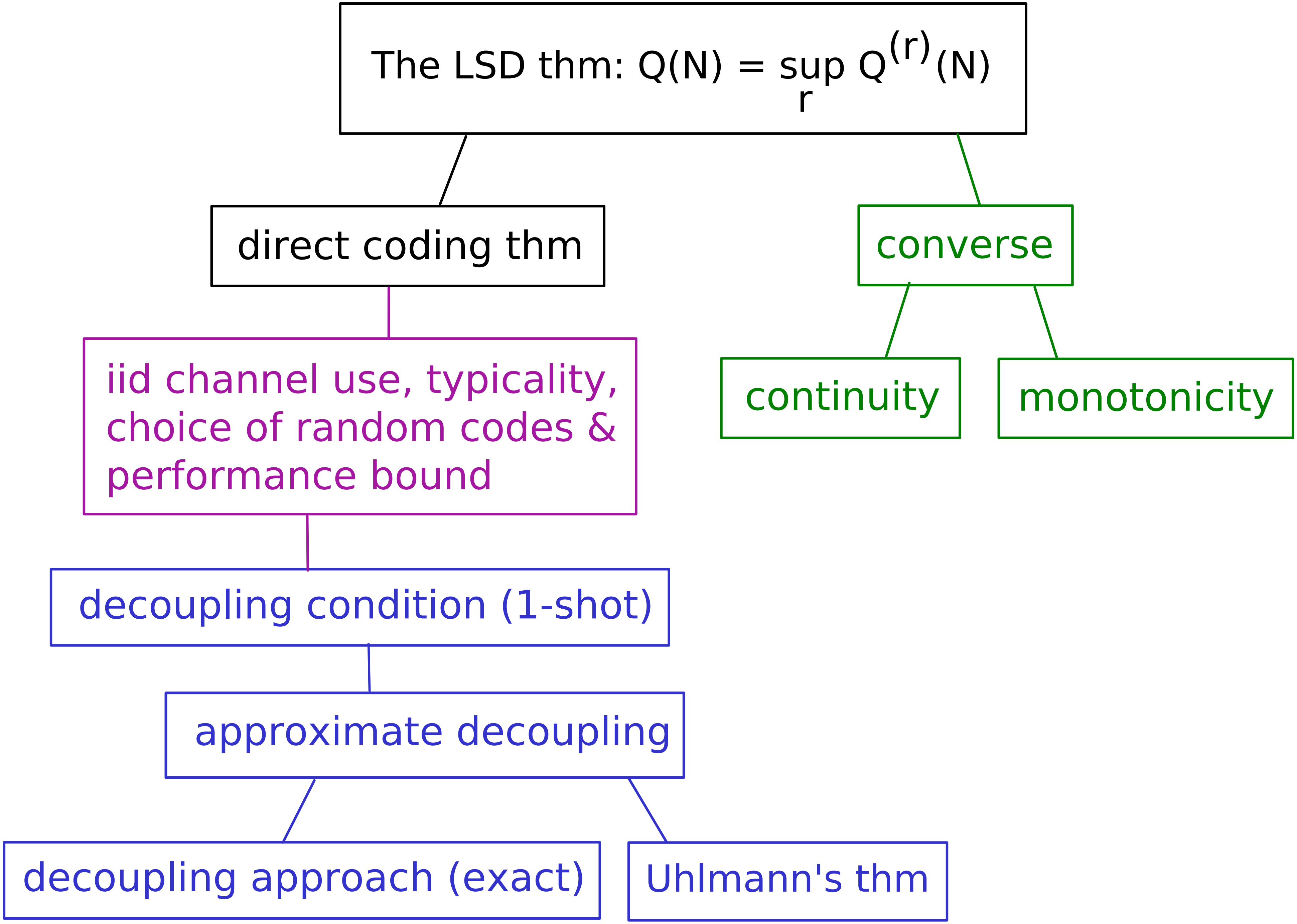
$|\eta^r\rangle$

a subspace of the typical space of $(\eta^r_{A^r})^{\otimes n}$ is the code space

can transmit ~ n * coherent info of $|\eta^r\rangle$

channel used nr times, so, overall rate $\frac{1}{r}$ coherent info of $|\eta^r\rangle$

optimize over r-use input, rate = r-shot coherent info of N



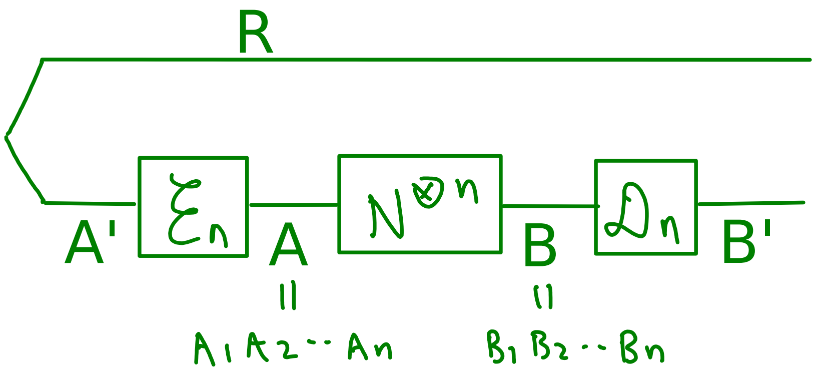
The LSD thm: Q(N) = sup_r Q^(r)(N)
direct coding thm
converse
iid channel use, typicality, choice of random codes & performance bound
continuity
monotonicity
decoupling condition (1-shot)
approximate decoupling
decoupling approach (exact)
Uhlmann's thm

Converse:

Suppose there is a sequence of (M,n) codes:

$$|\Phi_M\rangle = \frac{1}{\sqrt{M}}\sum_{i=1}^{M}|i\rangle|i\rangle$$

$$\Phi_M = |\Phi_M\rangle\langle\Phi_M|$$



s.t. $\left\| I\otimes(\mathcal{D}_n\circ N^{\otimes n}\circ\mathcal{E}_n)(\Phi_M) - \Phi_M\right\|_1 \le \epsilon_n \to 0$

Then, $Q^n(N) \ge \frac{1}{n} I_c(R\rangle B)_{I\otimes(N^{\otimes n}\circ\mathcal{E})(\Phi_M)}$

$\ge \frac{1}{n} I_c(R\rangle B')_{I\otimes(\mathcal{D}_n\circ N^{\otimes n}\circ\mathcal{E})(\Phi_M)}$ monotonicity under processing on 2nd sys

ϵ_n-close

$\ge \frac{1}{n}\left[I_c(R\rangle B')_{\Phi_M} + 4\epsilon_n\log M + 2h(\epsilon_n)\right]$ continuity

binary entropy function

$= \underbrace{\frac{1}{n}\left[\log M\right.}_{\text{rate}} + \underbrace{4\epsilon_n\log M + 2h(\epsilon_n)}_{\text{negligible as n grows and } \epsilon_n \to 0}\left.\right]$

$\therefore$ achievable rate $\leq Q^{(n)}(N) \leq \sup_{(n)} Q^{(n)}(N)$

$\therefore Q(N) \leq \sup_{(n)} Q^{(n)}(N)$

This completes the proof for the LSD theorem.

Next week:
Degradable channels, erasure channel
Nonadditivity of coherent information, depolarizing channel

Next next week:
Superactivation
Recent bounds

Mention correction to last lecture.