CO781 / QIC 890:

Theory of Quantum Communication

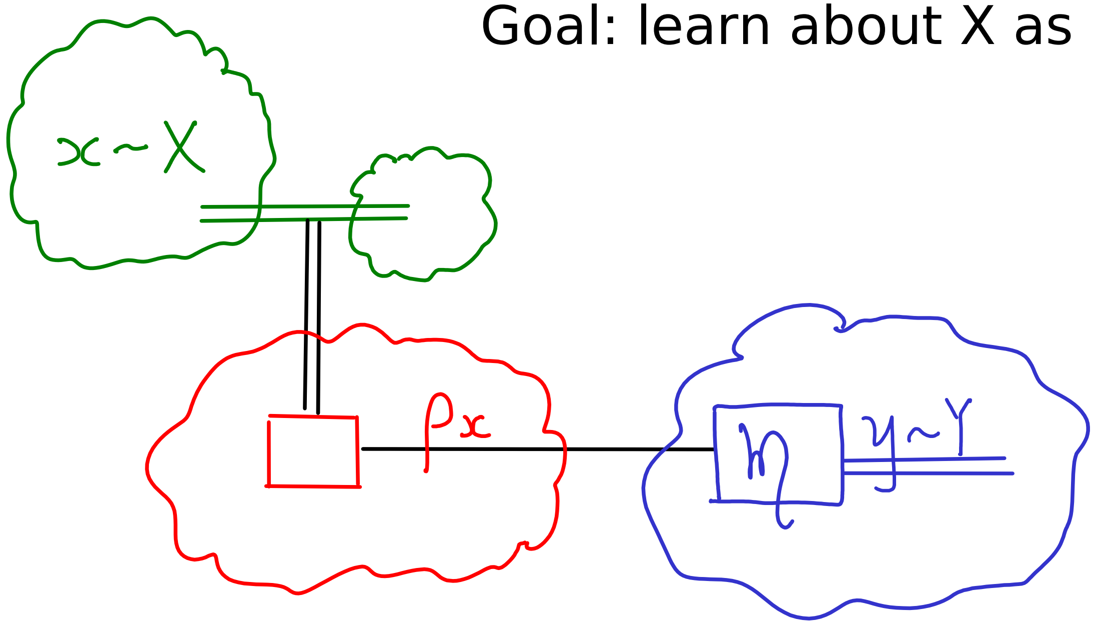
Topics 4, part 2

Encoding classical information in quantum states
and retrieving it

Scenario 1: accessible information

Scenarios of encoding classical data into quantum states & retrieving it

Goal: learn about X as much as possible via rv Y.



joint distribution p(xy)

e.g.

X: time elapsed	Atomic clock	Time reported
X: black hole	Telescopes generating squeezed states	Detector sees signals
X: oracles	Quantum circuit	meas giving comp answer
X: message	Alice's encoding map + noisy quantum channels	Bob's decoder giving decoded message

Scenarios: who controls each of the steps, measure of success ...

Scenario 0: no control throughout -- a draw of XY.

Scenarios of encoding classical data into quantum states & retrieving it

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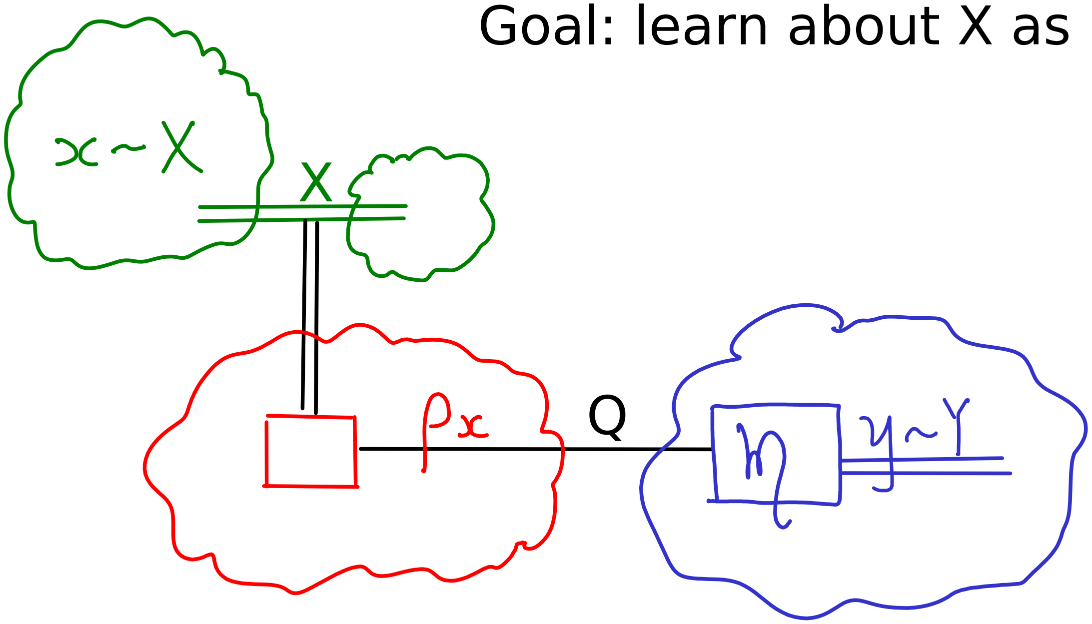

joint distribution p(xy)

Scenario 1: accessible information / states discrimination

p_x , ρ_x predetermined

Richard draws x with prob p(x), prepares ρ_x , gives state to Bob

Bob picks measurement

(a) max prob(X=Y): state discrimination

(b) max I(X:Y): accessible information

today

Scenarios of encoding classical data into quantum states & retrieving it

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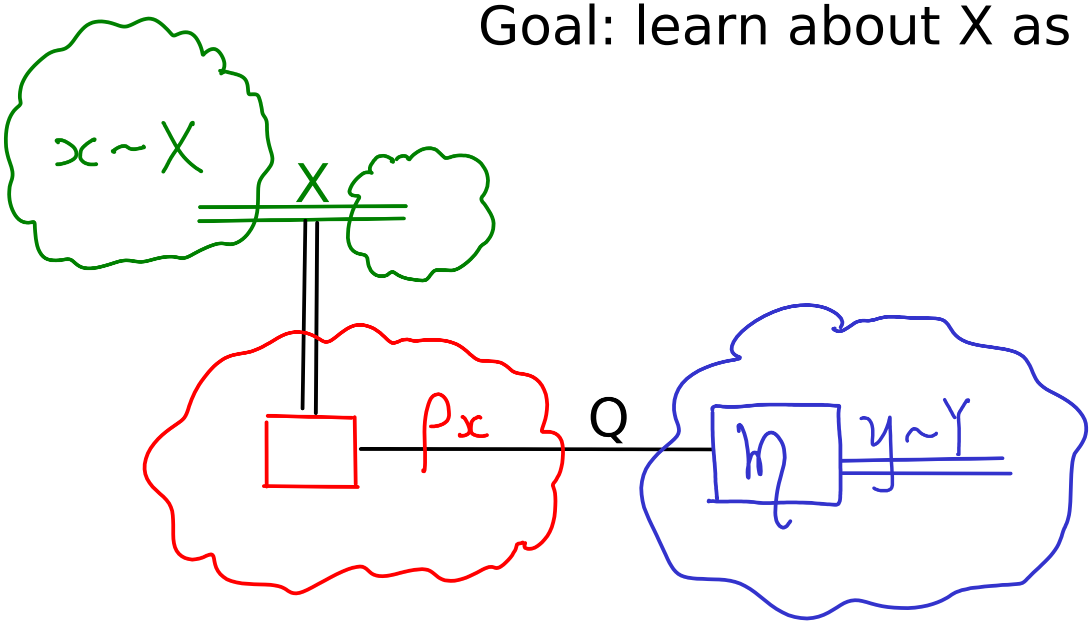


joint distribution p(xy)

Scenario 2: classical channel

ρ_x, $\mathcal{M}$ predetermined POVM $\{M_y\}$, $M_y \geq 0$, $\sum_y M_y = I$

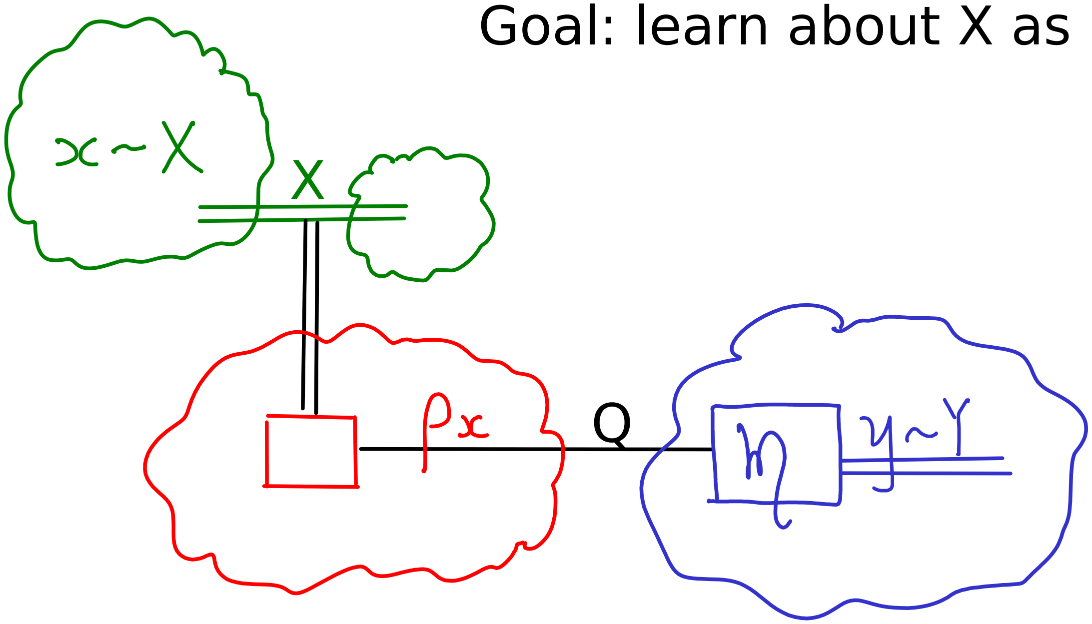
Alice chooses x,
corresponding state ρ_x generated, and measured (with fixed meas),
outcome is given to Bob.

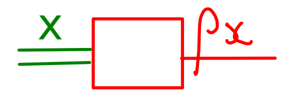
For each x, Bob receives y with prob p(y|x) = $\operatorname{tr} M_y \rho_x$

Last week: for large number of uses,
can create max_p(x) I(X:Y) cbits per use

Scenarios of encoding classical data into quantum states & retrieving it

Goal: learn about X as much as possible via rv Y.



Q box 

as if Alice presses a button "x" and Q box spits out ρ_x to Bob

Scenario 3: Q box

ρ_x predetermined

Alice chooses x,

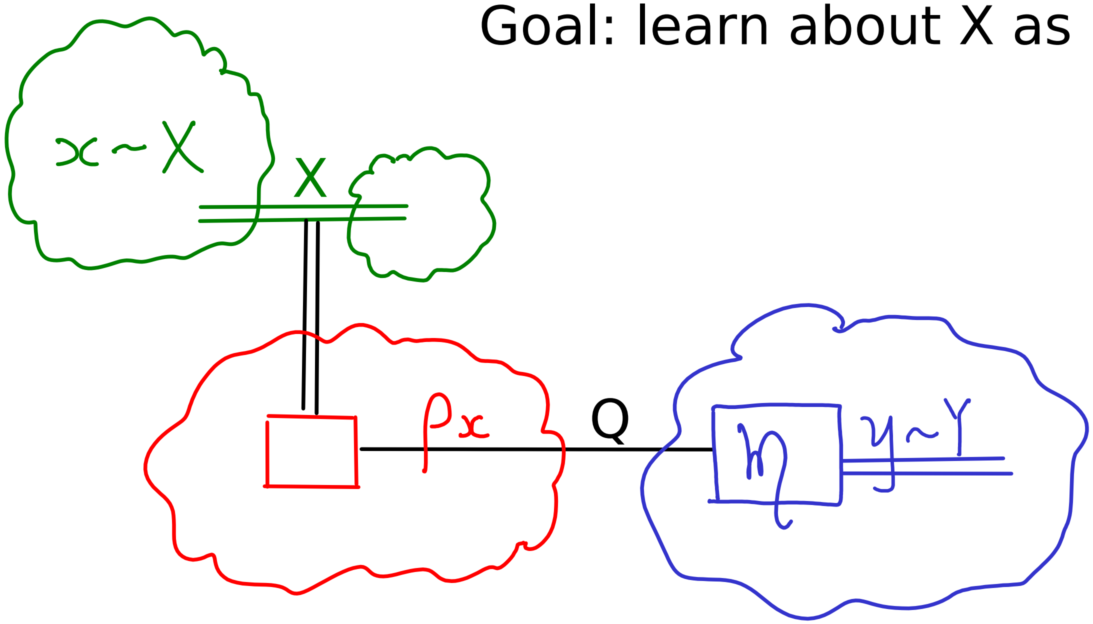
corresponding state ρ_x generated and available to Bob.

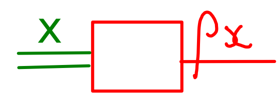
Bob picks measurement and obtains y.

If Bob sticks to optimal meas for I(X:Y) for each system, this reduces to scenario 2.

Scenarios of encoding classical data into quantum states & retrieving it

Goal: learn about X as much as possible via rv Y.



Q box 

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Scenario 3: Q box

ρ_x predetermined

Alice chooses x,

corresponding state ρ_x generated and available to Bob.

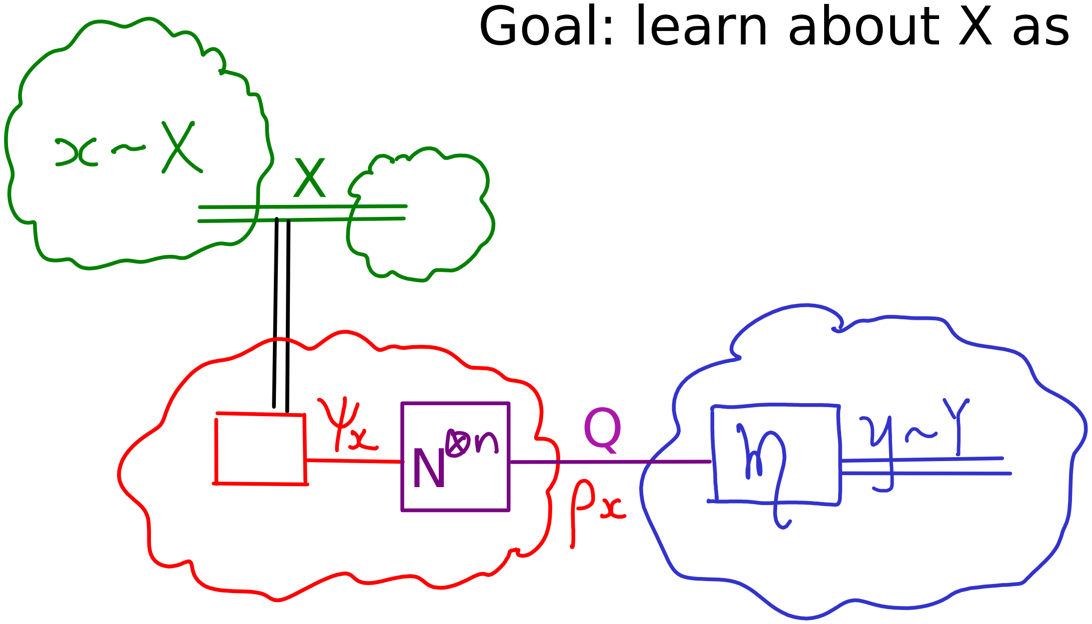
Bob picks measurement and obtains y.

Scenario 3: for multiple uses, Bob can choose JOINT measurement.

Next Tue: for large number of uses of Q boxes, can create S(X:Q) cbits per use, for $\Lambda = \sum_x p_x \, |x\rangle\langle x|_X \otimes \rho_{x\,Q}$.

Scenarios of encoding classical data into quantum states & retrieving it

Goal: learn about X as much as possible via rv Y.



Scenario 4: classical capacity of quantum channel given N

Alice chooses x, and ψ_x (the input to n uses of N)

ρ_x is the channel output available to Bob.

Bob picks measurement and obtains y.

Scenario 4: for multiple uses, Bob can choose JOINT measurement.

Optimized: C(N) classical capacity of quantum channel N (next Thur).

Definition:

Let $\Lambda = \sum_x p_x \, |x\rangle\langle x|_X \otimes \rho_{x\,Q}$.

$\mathcal{M}$ measurement on Q with output space Y

The accessible information for ensemble $\mathcal{E} = \{p_x, \rho_x\}$ is

$$I_{acc}(\mathcal{E}) := \max_{\mathcal{M}} I(X:Y)_{I \otimes \mathcal{M}(\Lambda)}$$

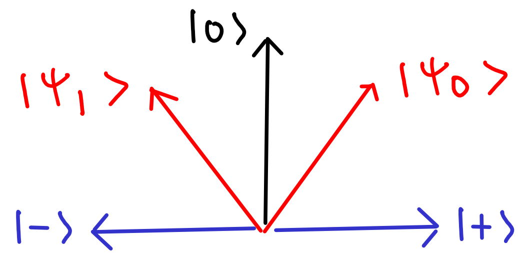
Example 1. $x = 0, 1$, $p(0) = p(1) = 1/2$,

$\rho_x = |\psi_x\rangle\langle\psi_x|$, $|\psi_0\rangle = a|0\rangle + b|1\rangle$

$|\psi_1\rangle = a|0\rangle - b|1\rangle$

$a, b \geq 0$, $a^2 + b^2 = 1$

(most general form of 2 arbitrary pure states)



Optimal measurement:
projective, along basis $\{|+\rangle, |-\rangle\}$

Levitin 95, or Fuchs PhD thesis 96 (Ch3.5)

$$p(x=0\ y=+) = \underbrace{p(x=0)}_{=\frac{1}{2}}\ \underbrace{p(y=+|x=0)}_{=\operatorname{tr}|+\rangle\langle+|\psi_0\rangle\langle\psi_0|}$$

$$\operatorname{tr}|+\rangle\langle+|\psi_0\rangle\langle\psi_0| = \left(\frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}}\right)^2 = \frac{1}{2}(a^2 + b^2 + 2ab) = \frac{1}{2} + ab$$

$$p(x=1\ y=+) = \frac{1}{2}\left(\frac{1}{2} - ab\right)$$

$p(y=+) = 1/2$,

$p(x=0|y=+) = 1/2 + ab$
$p(x=1|y=+) = 1/2 - ab$

$H(X|y=+) = h(1/2+ab)$

Similarly,
$p(y= -) = 1/2$,
$H(X|y= -) = h(1/2-ab) = h(1/2+ab)$

$H(X|Y) = p(y=+)\,H(X|y=+) + p(y=-)\,H(X|y=-) = h(1/2+ab)$

$I_{acc} = I(X:Y) = H(X) - H(X|Y) = 1 - h(1/2+ab)$

How to optimize measurement for Iacc?

1. Unknown for most ensembles

   For the few ensembles (highly symmetric) with known optimal measurements, there is no simple proof of optimality :(

<u>How to optimize measurement for Iacc?</u>

2. EB Davies, IEEE Trans Info Th, 24, p596, 1978

For any ensemble of states in d dimensions, $\mathcal{E} = \{p_x, \rho_x\}$
optimal measurement has POVM $\mathcal{M} = \{M_y\}_{y=1}^{n}$ with

(a) rank(M_y) = 1 and

(b) $d \leq n \leq d^2$

Proof (a): If $M_y = \sum_k M_{y,k}$ is a decomp into rank 1 matrices

replace measurement $\mathcal{M}$ with POVM $\{M_y\}$

by new measurement $\mathcal{M}'$ with POVM $\{M_{y,k}\}$. outcome has 2 parts

eg $\mathcal{M}$: $M_0 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{3}|+\rangle\langle +|$ $(y=0)$

$M_1 = \frac{1}{5}|1\rangle\langle 1| + \frac{1}{2}|-\rangle\langle -|$ $(y=1)$

$M_2 = \frac{4}{5}|1\rangle\langle 1| + \frac{1}{6}|+\rangle\langle +|$ $(y=2)$

$\mathcal{M}'$: $M_{0,0} = \frac{1}{2}|0\rangle\langle 0|$ $(y=0, k=0)$ $\quad M_{0,1} = \frac{1}{3}|+\rangle\langle +|$ $(y=0, k=1)$

$M_{1,0} = \frac{1}{5}|1\rangle\langle 1|$ $(y=1, k=0)$ $\quad M_{1,1} = \frac{1}{2}|-\rangle\langle -|$ $(y=1, k=1)$

$M_{2,0} = \frac{4}{5}|1\rangle\langle 1|$ $(y=2, k=0)$ $\quad M_{2,1} = \frac{1}{6}|+\rangle\langle +|$ $(y=2, k=1)$

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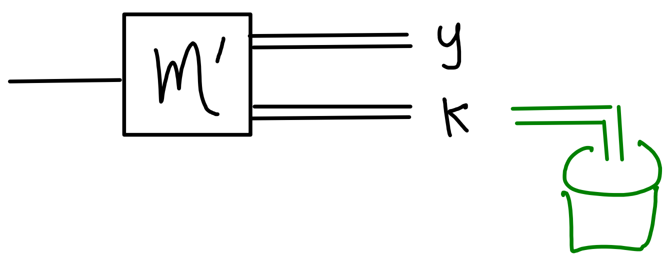

if we discard k, we perform $\mathcal{M}$

$\therefore$ Iacc $(\mathcal{E})$ $=$ $I(X:Y)_{I\otimes\mathcal{M}(\Lambda)}$ $=$ $I(X:Y)_{I\otimes\mathcal{M}'(\Lambda)}$ $\leqslant$ $I(X:YK)_{I\otimes\mathcal{M}'(\Lambda)}$ $\leqslant$ Iacc

given $\mathcal{M}$ optimal

$\Lambda = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$

q info processing ineq

equal and $\mathcal{M}'$ also optimal

How to optimize measurement for Iacc?

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Proof (b): see e.g., Watrous book, or 1904.10985 Corollary 5.

Based on:

Caratheodory's Theorem:

Let $S \subseteq \mathbb{R}^t$, conv(S) convex hull of S.

Then, any $x \in$ conv(S) is a convex combination of at most

t+1 elements of S.

How to optimize measurement for Iacc?

3. EB Davies, IEEE Trans Info Th, 24, p596, 1978
   Sasaki, Barnett, Jozsa, Osaki, Hirota 9812062
   Decker 0509122

   Informally: many equiprobable ensembles of states with symmetry have optimal measurement with the same symmetry.

How to optimize measurement for Iacc?

3. Ensembles with symmetry

Example 2. Define the ensemble $\mathcal{E}_1$ with

$p(0) = p(1) = p(2) = 1/3$, $\rho_x = |\psi_x\rangle\langle\psi_x|$, $|\psi_0\rangle = |0\rangle$

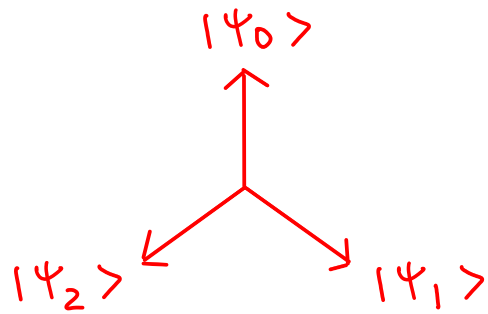
$|\psi_1\rangle = \cos\frac{\pi}{3}|0\rangle + \sin\frac{\pi}{3}|1\rangle$

$|\psi_2\rangle = \cos\frac{\pi}{3}|0\rangle - \sin\frac{\pi}{3}|1\rangle$

9812062: optimal meas has POVM

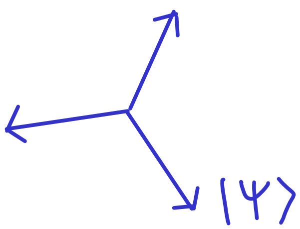
$M_1 = \{ M_k = \frac{2}{3} R^k |\psi\rangle\langle\psi| R^{k\dagger} \}_{k=0,1,2}$

where $R = e^{i\sigma_y \frac{2}{3}\pi}$ (note $R^k|\psi_0\rangle = |\psi_k\rangle$)



(the trine or "Mercedes" states)

e.g.,



Ex: optimal

$|\psi\rangle = |\psi_0^\perp\rangle = |1\rangle$.

So, $M_0 = |\psi_0^\perp\rangle\langle\psi_0^\perp| = |1\rangle\langle 1|$

$M_1 = |\psi_1^\perp\rangle\langle\psi_1^\perp|$, $|\psi_1^\perp\rangle = \sin\frac{\pi}{3}|0\rangle - \cos\frac{\pi}{3}|1\rangle$

$M_2 = |\psi_2^\perp\rangle\langle\psi_2^\perp|$, $|\psi_2^\perp\rangle = \sin\frac{\pi}{3}|0\rangle + \cos\frac{\pi}{3}|1\rangle$

How to optimize measurement for Iacc?

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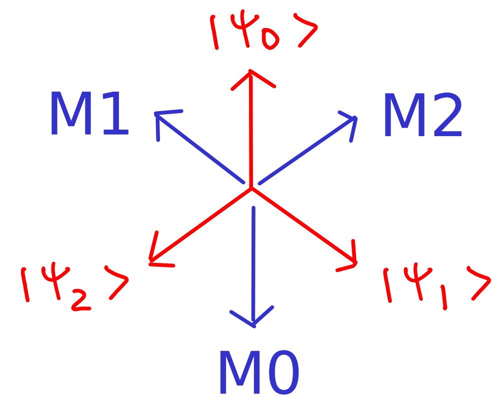
$|\psi_2\rangle = \cos\frac{\pi}{3}|0\rangle - \sin\frac{\pi}{3}|1\rangle$

$\mathcal{M}_1 = \{M_k\}_{k=0,1,2}$

$M_0 = |\psi_0^\perp\rangle\langle\psi_0^\perp| = |1\rangle\langle 1|$

$M_1 = |\psi_1^\perp\rangle\langle\psi_1^\perp|$, $|\psi_1^\perp\rangle = \sin\frac{\pi}{3}|0\rangle - \cos\frac{\pi}{3}|1\rangle$

$M_2 = |\psi_2^\perp\rangle\langle\psi_2^\perp|$, $|\psi_2^\perp\rangle = \sin\frac{\pi}{3}|0\rangle + \cos\frac{\pi}{3}|1\rangle$



Ex: find pr(y|x) for all x,y.

If y=0, pr(x=0|y=0) = 0

pr(x=1|y=0) = pr(x=2|y=0) = 1/2, so H(X|y=0) = 1.

$$H(X|Y) = \underbrace{p(y=0)}_{\frac{1}{3}} \cdot \underbrace{H(X|y=0)}_{1} + \underbrace{p(y=1)}_{\frac{1}{3}} \cdot \underbrace{H(X|y=1)}_{1} + \underbrace{p(y=2)}_{\frac{1}{3}} \cdot \underbrace{H(X|y=2)}_{1} = 1$$

Iacc = H(X) - H(X|Y) = (log 3) - 1 = 0.5850.

How to optimize measurement for Iacc?

4. Additivity of accessible info on product ensembles

Let $\mathcal{F}_1 = \{p(x_1), \rho_{x_1}\}$, $\mathcal{F}_2 = \{p(x_2), \sigma_{x_2}\}$

The product ensemble of $\mathcal{F}_1, \mathcal{F}_2$ is

$$\mathcal{F}_1 \otimes \mathcal{F}_2 = \{p(x_1)\,p(x_2),\ \rho_{x_1} \otimes \sigma_{x_2}\}$$

Represent $\mathcal{F}_1$ by $\Lambda_1 = \sum_{x_1} p(x_1)\,|x_1\rangle\langle x_1| \otimes \rho_{x_1}$

$\mathcal{F}_2$ by $\Lambda_2 = \sum_{x_2} p(x_2)\,|x_2\rangle\langle x_2| \otimes \sigma_{x_2}$

$\mathcal{F}_1 \otimes \mathcal{F}_2$ by $\sum_{x_1 x_2} p(x_1)\,p(x_2)\,\underbrace{|x_1\rangle\langle x_1|}_{X_1} \otimes \underbrace{|x_2\rangle\langle x_2|}_{X_2} \otimes \underbrace{\rho_{x_1}}_{Q_1} \otimes \underbrace{\sigma_{x_2}}_{Q_2} = \Lambda_1 \otimes \Lambda_2$

Thm. $I_{acc}(\mathcal{F}_1 \otimes \mathcal{F}_2) = I_{acc}(\mathcal{F}_1) + I_{acc}(\mathcal{F}_2)$

joint meas allowed but not needed

Idea: applying the optimal measurement of $\mathcal{F}_1$ on Q1,
and the optimal measurement of $\mathcal{F}_2$ on Q2,
is optimal for $\mathcal{F}_1 \otimes \mathcal{F}_2$

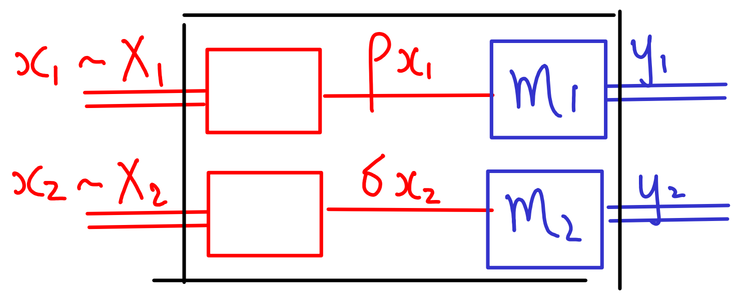
Thm. $I_{acc}(\mathcal{F}_1 \otimes \mathcal{F}_2) = I_{acc}(\mathcal{F}_1) + I_{acc}(\mathcal{F}_2)$

Proof: [$\geq$] suffices to find a measurement for $\mathcal{F}_1 \otimes \mathcal{F}_2$ achieving mutual information given by RHS.

Let $\mathcal{M}_1, \mathcal{M}_2$ be optimal meas for $\mathcal{F}_1, \mathcal{F}_2$ with outputs Y_1, Y_2.

We now analyse the mutual info between X1X2 and Y1Y2

if $\mathcal{M}_1 \otimes \mathcal{M}_2$ is applied to Q1Q2 of ensemble $\mathcal{F}_1 \otimes \mathcal{F}_2$



$I_{acc}(\mathcal{F}_1 \otimes \mathcal{F}_2) \geq$ I(X1 X2 : Y1 Y2) $_{I \otimes I \otimes \mathcal{M}_1 \otimes \mathcal{M}_2(\Lambda)}$

X1 independent of X2
Y1 independent of Y2
X1Y1 independent of X2Y2

= H(X1 X2) + H(Y1 Y2) - H(X1 X2 Y1 Y2)

= H(X1) + H(X2) + H(Y1) + H(Y2) - H(X1 Y1) - H(Y1 Y2)

= [H(X1) + H(Y1) - H(X1 Y1)] $_{I \otimes \mathcal{M}_1(\Lambda_1)}$
+ [H(X2) + H(Y2) - H(Y1 Y2)] $_{I \otimes \mathcal{M}_2(\Lambda_2)}$

reduced states of $I \otimes I \otimes \mathcal{M}_1 \otimes \mathcal{M}_2 \, (\Lambda = \Lambda_1 \otimes \Lambda_2)$

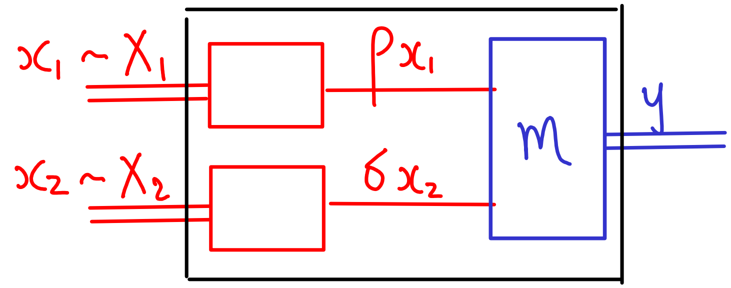
= I(X1:Y1) $_{I \otimes \mathcal{M}_1(\Lambda_1)}$ + I(X2:Y2) $_{I \otimes \mathcal{M}_2(\Lambda_2)}$

$= I_{acc}(\mathcal{F}_1) + I_{acc}(\mathcal{F}_2)$ (by optimality of $\mathcal{M}_1, \mathcal{M}_2$)

Thm. $I_{acc}(\mathcal{F}_1 \otimes \mathcal{F}_2) = I_{acc}(\mathcal{F}_1) + I_{acc}(\mathcal{F}_2)$

Proof: [$\leq$]

Let $\mathcal{M}$ be any meas on Q1 Q2
with output space Y



I(X1 X2 : Y) = I(X1:Y) + I(X2:Y|X1)

H(X1X2) + H(Y)
-H(X1 X2 Y)

H(X1) + H(Y)
-H(X1 Y)

H(X2|X1) - H(X2|X1Y)
= H(X1X2) - H(X1) - H(X2X1Y) + H(X1 Y)

Thm. $I_{acc}(\mathcal{F}_1 \otimes \mathcal{F}_2) = I_{acc}(\mathcal{F}_1) + I_{acc}(\mathcal{F}_2)$

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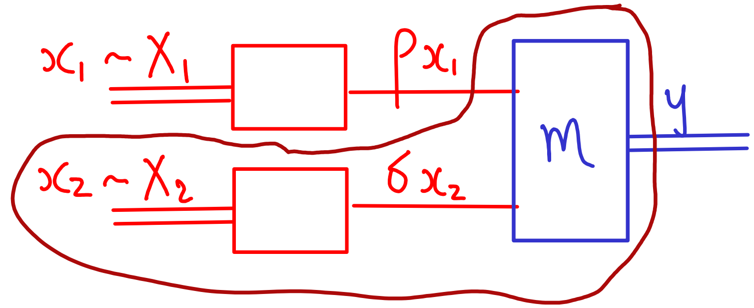

I(X1 X2 : Y) = I(X1:Y) + I(X2:Y|X1)

H(X1X2) + H(Y)
-H(X1 X2 Y)

H(X1) + H(Y)
-H(X1 Y)

H(X2|X1) - H(X2|X1Y)
= H(X1X2) - H(X1) - H(X2X1Y) + H(X1 Y)

(1) if X1, X2 independent,

drawing state from $\mathcal{F}_2$
is part of meas of Q1

is a measurement on Q1 prepared in the ensemble $\mathcal{F}_1$

$\therefore$ I(X1:Y) $\leq I_{acc}(\mathcal{F}_1)$

(2) 2nd term $\sum_{x_1} p(x_1)\, I(X_2 : Y \mid X_1 = x_1) \leq \max_{x_1} I(X_2 : Y \mid X_1 = x_1) \leq I_{acc}(\mathcal{F}_2)$

prepare best ρ_{x_1} as
part of meas of Q2

$\therefore \forall \mathcal{M}.\quad I(X_1 X_2 : Y) \leq I_{acc}(\mathcal{F}_1) + I_{acc}(\mathcal{F}_2)$

$\therefore\ I_{acc}(\mathcal{F}_1 \otimes \mathcal{F}_2) \leq I_{acc}(\mathcal{F}_1) + I_{acc}(\mathcal{F}_2)$

Tue

adaptive, vs joint A3

locking A3

upper bound by holevo info

holevo bound

araki lieb two-way holevo bound