CO781 / QIC 890:

Theory of Quantum Communication

Topic 2, part 3

The asymptotic equipartition theorem,
Shannon entropy and classical data compression

von Neumann entropy, Quantum data compression,

Entanglement concentration and dilution
Entropy of entanglement
Entanglement spread
Embezzlement of entanglement

References:

Preskill Sections 10.1.1, 10.3, 10.4

Entanglement concentration & dilution,
entropy of entanglement

arXiv: quant-ph

Bennett, Bernstein, Popescu, Schumacher, 9511030

Lo, Popescu, 9902045

Lower bound for dilution & entanglement spread

Harrow, Lo, 0204096, Hayden, Winter, 0204092

Harrow, 0909.1557

Embezzlement:

Hayden, van Dam 0201041

Leung, Toner, Watrous, 0804.4118

Consider the task:

For a given state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$

(known to all of Richard, Alice, Bob)

Richard prepares $|\psi\rangle^{\otimes n}$ on R1A1, R2A2, ..., RnAn

gives A1 A2 ... An to Alice

Alice encodes A1 ... An into nr qubits

Alice sends those nr qubits to Bob

Bob decodes those nr qubits, output B1 B2 ... Bn.

Requires final state on R1 B1 R2 B2 ... Rn Bn $\approx |\psi\rangle^{\otimes n}$

How to min r ?

Let $\rho = \mathrm{tr}_R |\psi\rangle\langle\psi|$. TTS works, achieves the rate $r = S(\rho)$.

defined using ρ, n, $\delta = r - S(\rho)$, ϵ ....

Detail: use Schmidt decomposition $|\psi\rangle = \sum_{v} \sqrt{p_v}\, |f_v\rangle_R\, |e_v\rangle_A$

$$\rho = \sum_v p_v |e_v\rangle\langle e_v|$$ is a spectral decomposition.

Let $\epsilon > 0$, $\delta = r - S(\rho) > 0$, $T_{n,\delta}$ defined as before (on V).

$$\Pi_S = \sum_{v^n \in T_{n,\delta}} |e_{v^n}\rangle\langle e_{v^n}|$$

Apply TTS to A1 A2 ... An, output (on R1 B1 ... Rn Bn) =

$$\rho_{out} = \underset{R^n}{I} \otimes \underset{A^n}{\Pi_S} \underbrace{(|\psi\rangle\langle\psi|)^{\otimes n}}_{\text{on } R^n A^n} I \otimes \Pi_S + \mathrm{tr}\left(I \otimes (I - \Pi_S)(|\psi\rangle\langle\psi|)^{\otimes n}\right) \cdot \mathrm{ERR}$$

$$|\psi\rangle^{\otimes n} = \sum_{v^n} \sqrt{p_{v^n}}\, |f_{v^n}\rangle_{R^n} |e_{v^n}\rangle_{A^n}$$

$$(I \otimes \Pi_S)|\psi\rangle^{\otimes n} = \sum_{v^n \in T_{n,\delta}} \sqrt{p_{v^n}}\, |f_{v^n}\rangle_{R^n} |e_{v^n}\rangle_{A^n}$$

$$\langle\psi|^{\otimes n} (I \otimes \Pi_S) |\psi\rangle^{\otimes n} = \sum_{v^n \in T_{n,\delta}} p_{v^n} \geq 1 - \epsilon$$

$$\mathrm{tr}\left(I \otimes (I - \Pi_S)(|\psi\rangle\langle\psi|)^{\otimes n}\right) \leq \epsilon$$

$$\left\| \left(|\psi\rangle\langle\psi|\right)^{\otimes n} - \rho_{out} \right\|_1$$

$$= \left\| \left(|\psi\rangle\langle\psi|\right)^{\otimes n} - I\otimes\Pi_s \left(|\psi\rangle\langle\psi|\right)^{\otimes n} I\otimes\Pi_s - \mathrm{tr}\left(I\otimes(I-\Pi_s)\left(|\psi\rangle\langle\psi|\right)^{\otimes n}\right)\cdot ERR \right\|_1$$

$$\leq \left\| \left(|\psi\rangle\langle\psi|\right)^{\otimes n} - I\otimes\Pi_s \left(|\psi\rangle\langle\psi|\right)^{\otimes n} I\otimes\Pi_s \right\|_1 + \left\| \mathrm{tr}\left(I\otimes(I-\Pi_s)\left(|\psi\rangle\langle\psi|\right)^{\otimes n}\right)\cdot ERR \right\|_1$$

$$\lesssim 2\sqrt{1-\left(\langle\psi|^{\otimes n}\,(I\otimes\Pi_s)\,|\psi\rangle^{\otimes n}\right)^2} + \epsilon \leq 2\sqrt{2}\sqrt{\epsilon} + \epsilon.$$

notes on the detail last lecture ...

Entanglement dilution and concentration (BBPS9511030)

Will see later why classical comm cannot create entanglement

LOCC is the class of operations consisting of unlimited
Local Operations and Classical Communication

Two related questions in bipartite LOCC:

(a) How many ebits are needed to create approx of $|\psi\rangle^{\otimes n}$?
(b) How many ebits can be approx extracted from $|\psi\rangle^{\otimes n}$?

Approx in trace distance (for pure states, same as fidelity up to square root).

Task (a) is called entanglement dilution
Task (b) is called entanglement concentration

Answer for (a) upper bounds answer for (b).

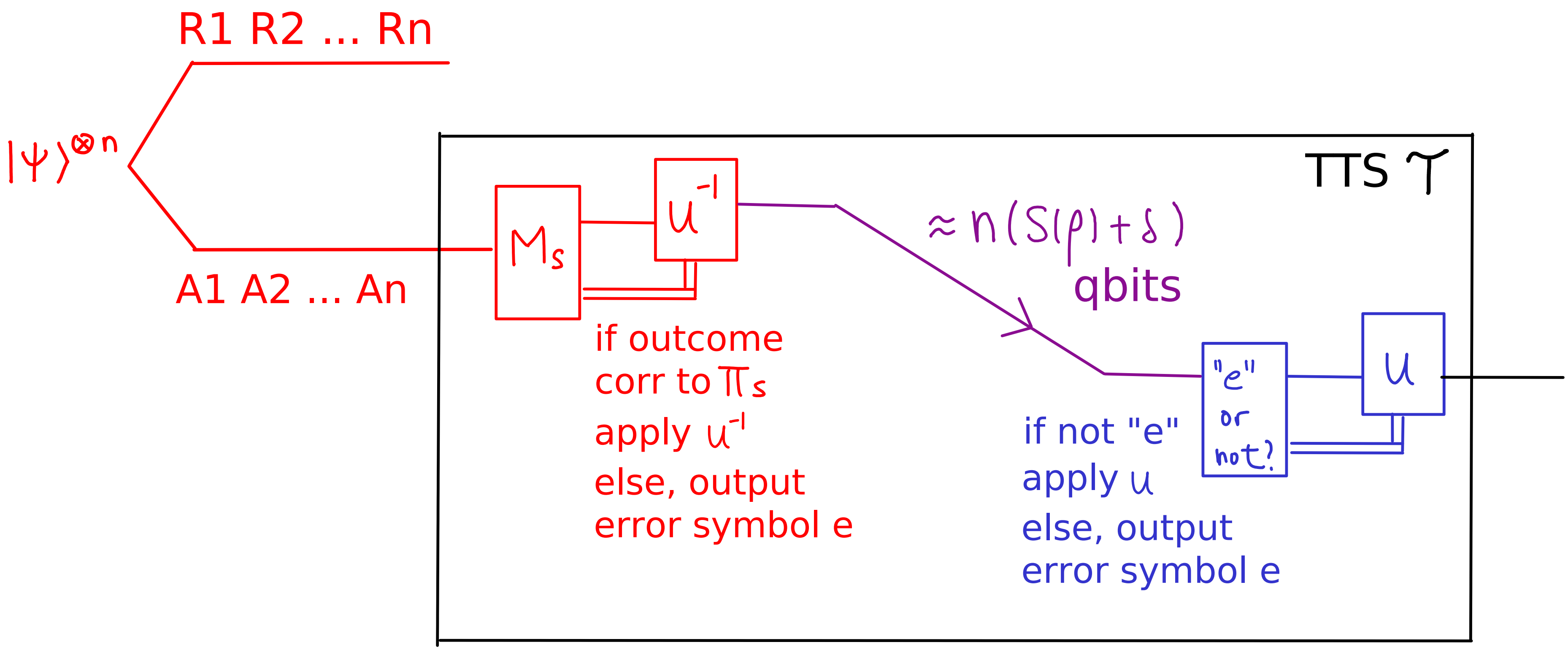
Nice surprise: both ans are $nS(\rho) + o(n)$, $\rho = \mathrm{tr}_B |\psi\rangle\langle\psi|$.

(Note both Alice and Bob know what states they are
transforming to and from.)

For entanglement dilution:

Simplest method: use the protocol to distribute $|\psi\rangle^{\otimes n}$ but obtain the qbits needed by teleportation.

In detail: here Alice is also Richard. She prepares $|\psi\rangle^{\otimes n}$.



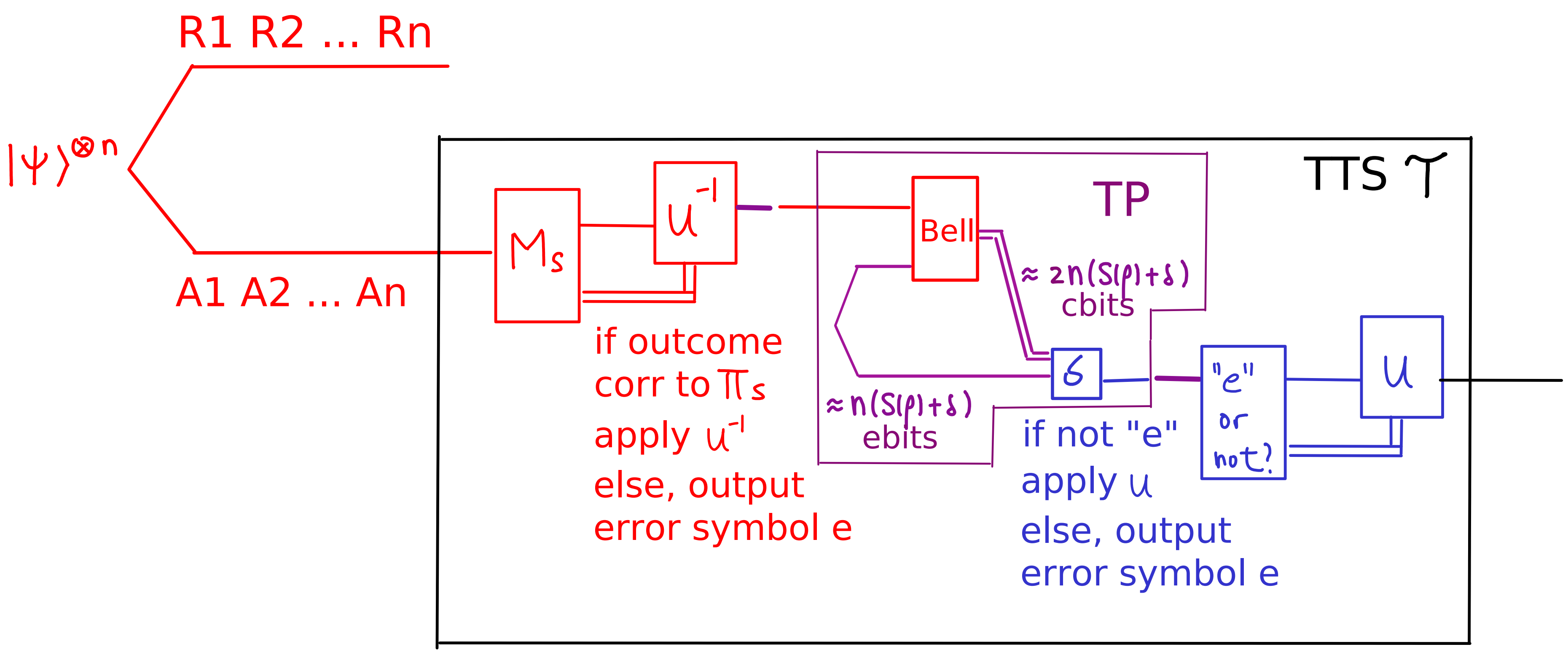
We already proved the above outputs a state close to $|\psi\rangle^{\otimes n}$.
Now replace qbit by TP (does't change the output).

For entanglement dilution:

Simplest method: $n(S(\rho)+\delta)$ ebits + $2n(S(\rho)+\delta)$ cbits $\geq |\psi\rangle^{\otimes n}$.

free in LOCC

In detail: here Alice is also Richard. She prepares $|\psi\rangle^{\otimes n}$.



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How to reduce the classical comm cost for dilution?

Main idea (Lo Popescu) (detail in A2):

up to local unitaries

$$|\psi\rangle^{\otimes n} \approx a \left(\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)\right)^{\otimes nS(\rho) - O(\sqrt{n})} \otimes |\mu\rangle + b\,|\nu\rangle$$

$O(\sqrt{n})$ qubit state

small

Method:

goes away when $b\,|\nu\rangle$ ignored

use $nS(\rho) - O(\sqrt{n})$ ebits

teleport using $O(\sqrt{n})$ ebits $O(\sqrt{n})$ cbits

ignore

For entanglement concentration:

Let $|\psi\rangle = \sum_{v} \sqrt{p_v}\, |e_v\rangle_A |f_v\rangle_B$

Local unitaries are free under LOCC, so, Alice and Bob first convert the Schmidt basis to computational basis for each copy. They now have $|\phi\rangle^{\otimes n}$ where

$$|\phi\rangle = \sum_{v} \sqrt{p_v}\, |v\rangle_A |v\rangle_B$$

$$|\phi\rangle^{\otimes n} = \sum_{v^n} \sqrt{p_{v^n}}\, |v^n\rangle_{A^n} |v^n\rangle_{B^n}$$

Tempting idea: each of Alice & Bob applies meas w/ POVM $\{\Pi_S, I-\Pi_S\}$. With prob $\geq 1-\epsilon$, obtain state

$$|\widetilde{\phi^n}\rangle \propto \sum_{v^n \in T_{n,\delta}} \sqrt{p_{v^n}}\, |v^n\rangle_{A^n} |v^n\rangle_{B^n}$$

While $2^{-n(S(\rho)+\delta)} \leq p_{v^n} \leq 2^{-n(S(\rho)-\delta)}$ (p_{v^n}'s roughly equal)

A2: $|\widetilde{\phi^n}\rangle$ is NOT close to being maximally entangled :(

For entanglement concentration:

Alice and Bob will make a much finer measurement.

Def: Let rv X has sample space {1,2,...,m}.

Let $x^n = x_1 x_2 \cdots x_n$ be the outcome for n iid draws.

Suppose $t_k = \# x_i$'s equal to k.

Then, x^n is in the type class $(t_1, t_2, \ldots, t_m)$.

m-tuples of non-neg integers summing to n

e.g., n coin tosses (m=2), all outcomes with
k 1's and (n-k) 2's are in the type class (k,n-k).

e.g., 20 throws of a dice (n=20, m=6).
Outcome 443265111564314622246 has

t1 = 3 , t2 = 4 , t3 = 2, t4 = 5, t5 = 2, t6 = 4

so, outcome is in the type class (3,4,2,5,2,4).

Outcomes in the same type class are exactly equiprobable.

For entanglement concentration:

Alice and Bob each measures the type class
(Alice measures A1 A2 ,,, An, Bob B1 B2 ... Bn).

They always get the same outcome.

Conditioned on each outcome, their postmeas state is maximally entangled, with Schmidt rank depending on which type class.

In A2, you will show that the expected # ebits is

$$n S(\rho) - o(n)$$

for $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$.

Idea holds for general d, if n large enough.

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(1) Operational meaning:

for bipartite pure entangled state in many copies

- LOCC conversion is approx reversible
- single "currency" (ebit) of entanglement

This gives meaning to the quantity $S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$ as "the amount of entanglement" in the state $|\psi\rangle$.

(2) Even better, above holds even if CC is charged

- concentration requires no CC
- dilution requires $\Theta(\sqrt{n})$ cbits

achievability: Lo-Popescu, necessity: Harrow-Lo-Hayden-Winter

(3) For bipartite pure state single copy, (1)-(2) don't hold.

LOCC conversion: Lo-Popescu 9703038

Nielsen 9811053 (majorization)

(4) For bipartite mixed state, many copies

Task (a) has no name (?). # ebits per copy required:

entanglement of formation Bennett DiVincenzo Smolin Wootters 96

regularized to "entanglement cost" Hayden (M) Horodecki Terhal 0008134

and the two can be different

Shor 0305035, Hastings 0809.3972

restricting to vanishing CC, # ebits per copy is called the entanglement of purification. Terhal Horodecki Leung DiVincenzo 0202044

Task (b) is called entanglement distillation

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ebits extracted per copy: distillable entanglement

Mix state has "noise" ... to be removed by distillation.

Distillation (with 1- or 2-way CC) is mathematically equivalent to noisy channel coding for sending quantum data through noisy quantum channels (+crypto apps)!

Also, distillable entanglement can be strictly smaller than entanglement cost for some state (e.g., "bound entangled states" have 0 distillable entanglement but positive entanglement cost). M, P, R Horodecki 9801069

So, no single entanglement measure for mixed state, and LOCC conversion can be irreversible.

(5) For 3 or more parties, pure state, large # of copies

no comparable conversion theory
many types of incomparable entanglement

Bennett Popescu Rohrlich Smolin Thapliyal 9908073

Entanglement spread:

Def: for bipartite state $|\psi\rangle_{AB}$, $\rho = \mathrm{tr}_B |\psi\rangle\langle\psi|$,

* its entanglement spread is defined as

$$\Delta(|\psi\rangle) = \log(\mathrm{rank}(\rho)) - \log \frac{1}{\|\rho\|_\infty}$$

* its ϵ perturbed entanglement spread is defined as

$$\Delta_\epsilon(|\psi\rangle) = \min_{\substack{P:\text{ projectors} \\ \mathrm{tr}(\rho P) \geq 1-\epsilon}} \Delta(P \otimes I |\psi\rangle)$$

e.g., $\Delta = 0$ iff all nonzero Schmidt coeffs are equal.

The transformation: $|\phi\rangle \to |\tilde{\psi}\rangle$

s.t. fidelity of $|\psi\rangle, |\tilde{\psi}\rangle \geq 1-\epsilon$

requires $C \geq \Delta_{(4\epsilon)^{\frac{1}{8}}}(|\psi\rangle) - \Delta_0(|\phi\rangle) + 2\log(1-(4\epsilon)^{\frac{1}{8}})$ cbits.

i.e., increase in spread must be "paid for" by classical communication.

For entanglement dilution:

$$|\Phi\rangle^{\otimes nr} \longrightarrow |\psi\rangle^{\otimes n}$$

ebits

0 spread

$O(n)$ spread

$\Theta(\sqrt{n})$ ϵ perturbed spread

lower bound of cbits needed