Theorem: superdense coding (Bennett-Wiesner 93)

Suppose Alice and Bob share the state $\frac{1}{\sqrt{s}} \sum_{i=1}^{s} |i\rangle \otimes |i\rangle$ and Alice can send an $s$-dimensional quantum system to Bob. Then, Alice can communicate $t = s^2$ messages to Bob!
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How to think about quantum protocols:
Which party has what classical information?

Which party has what quantum system?

What operations he/she is allowed to do?
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How to think about quantum protocols:

Which party has what classical information?

Alice has a message $v \in \{0,x,y,z\}$. Bob has nothing.

Which party has what quantum system?

Initially, Alice (Bob) has the 1st register A (B) of the shared state. Alice also has another $s$-dim system C. She sends C to Bob. Then, Bob has both B and C.
Theorem: superdense coding (Bennett-Wiesner 93)

Suppose Alice and Bob share the state \( \frac{1}{\sqrt{s}} \sum_{i=1}^{s} |i\rangle \otimes |i\rangle \) and Alice can send an \( s \)-dimensional quantum system to Bob. Then, Alice can communicate \( t = s^2 \) messages to Bob!

How to think about quantum protocols:

What operations he/she is allowed to do?

Before Alice sends \( C \) to Bob, she can apply any operation on \( AC \) that depends on \( v \). \( C \) depends on \( A \) and \( v \), and \( C \) can be \( A \) itself.

After Bob receives \( C \) from Alice, he can apply any operation on \( AC \) that does not depend on \( v \).
Proof: for simplicity, first consider $s=2$.
Suppose Alice & Bob share the state
\[ |\Phi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \]
so that Alice (Bob) holds the first (second) qubit A (B).
Proof: for simplicity, first consider $s=2$.
Suppose Alice & Bob share the state $|\Phi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$ so that Alice (Bob) holds the first (second) qubit $A$ ($B$).

Recall the Pauli matrices:
\[ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
Proof: for simplicity, first consider $s=2$.
Suppose Alice & Bob share the state $|\Phi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$ so that Alice (Bob) holds the first (second) qubit.

Recall the Pauli matrices:

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Suppose Alice wants to communicate a message $v$ from the set $\{0, x, y, z\}$.

If her message is $v$, she applies $\sigma_v$ to $A$.

The shared state $|\Phi_0\rangle$ on $AB$ is transformed by $\sigma_v \otimes I$. 
These 4 states are mutually orthogonal, forming the "Bell basis". Note that Alice operates on a 2-dim system A, but the shared state on AB traverses to 1 out of 4 possible distinguishable (ortho) states.

For \( |\Phi_o\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \)

\[ 6_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 6_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad 6_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad 6_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ |\Phi_o\rangle = 6_o \otimes I \quad |\Phi_o\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \]

\[ |\Phi_x\rangle = 6_x \otimes I \quad |\Phi_o\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle) \]

\[ |\Phi_y\rangle = 6_y \otimes I \quad |\Phi_o\rangle = \frac{1}{\sqrt{2}} (|1\rangle |0\rangle - i |0\rangle |1\rangle) \]

\[ |\Phi_z\rangle = 6_z \otimes I \quad |\Phi_o\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \]
For \( |\Phi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \)

\[ \begin{align*} 
\mathbf{e}_0 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \mathbf{e}_x &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \mathbf{e}_y &= \begin{pmatrix} 0 \\ i \end{pmatrix}, & \mathbf{e}_z &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} 
\end{align*} \]

\( |\Phi_0\rangle = 6_0 \otimes \mathbb{I} \) \( |\Phi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \)

\( |\Phi_x\rangle = 6_x \otimes \mathbb{I} \) \( |\Phi_0\rangle = \frac{1}{\sqrt{2}} (|11\rangle + |10\rangle) \)

\( |\Phi_y\rangle = 6_y \otimes \mathbb{I} \) \( |\Phi_0\rangle = \frac{1}{\sqrt{2}} (|01\rangle - i|10\rangle) \)

\( |\Phi_z\rangle = 6_z \otimes \mathbb{I} \) \( |\Phi_0\rangle = \frac{1}{\sqrt{2}} (|10\rangle - |11\rangle) \)

These 4 states are mutually orthogonal, forming the "Bell basis". Note that Alice operates on a 2-dim system A, but the shared state on AB tranverses to 1 out of 4 possible distinguishable (ortho) states.

If Alice sends C=A to Bob, he has AB in the state \( |\Phi_v\rangle \).
He can measure AB along the Bell basis to find \( v \)!
Communication protocol:

|Φ₀⟩

1. Initial state shared between Alice and Bob. Alice is holding system A; Bob is holding system B.
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Communication protocol:

1. Initial state shared between Alice and Bob. Alice is holding system A; Bob is holding system B.

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3. Alice sends system A to Bob (2-dim).
Initial state shared between Alice and Bob. Alice is holding system A; Bob is holding system B.

If Alice wants to communicate \( \epsilon \{0, x, y, z\} \) to Bob she applies \( \mathcal{E}_v \) to qubit A. (4 possibilities)

Having both systems A & B, Bob measures along the Bell basis. Outcome is \( v \) with certainty.

Alice sends system C=A to Bob (2-dim).
Thoughts:

1. Entanglement enables the operation on a 2-dim system to map the shared state over 4 dimensions.

2. Bob has a 4-dim system (AB) after the channel transmission, so superdense coding is consistent with Holevo's bound.

3. Is there a catch? Does Alice also need to prepare the entangled state in AB and send B to Bob before superdense coding so altogether she sends 4 dims?

Not really. Bob can prepare the entangled state in AB and send A to Alice instead, or a common friend Charlie can prepare the entangled state and send A to Alice and B to Bob.

SD turns entanglement or back quantum comm into increased forward classical communication !!
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Suppose Alice and Bob share the state $\frac{1}{\sqrt{s}} \sum_{i=1}^{s} |i\rangle \otimes |i\rangle$ and Alice can send an $s$-dimensional quantum system to Bob. Then, Alice can communicate $t = s^2$ messages to Bob!

Converting the units of various resources:

$s$-dim quantum state = $\log s$ qubits
$s^2$ classical messages = $2 \log s$ bits
max entangled state of local dim $s = \log s$ "ebits"

$$\frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right)_{A_1 B_1} \otimes \cdots \otimes \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right)_{A_n B_n} = \frac{1}{\sqrt{2^n}} \sum_{u \in \{0,1\}^n} |u\rangle \otimes |u\rangle$$

Dividing everything by $\log s$, on average, SD coding uses 1 ebit and sends 1 qubit to communicate 2 bits (doubling the rate).
What if Alice wants to communicate a quantum state to Bob by sending only classical data?

For simplicity, she wants to communicate a qubit $|\psi\rangle = a|0\rangle + b|1\rangle$ to Bob.

Case (i): Alice knows $a,b$ (she authors the message)
She can send approximations of $a$ and $b$ to Bob. For Bob to decode a qubit closer and closer to $|\psi\rangle$ she has to send more and more bits.

Case (ii): Alice is given the state to be communicated (she runs Qedex, usual setting)
She does not know $a,b$, and cannot know more than 1 bit of information about them by Holevo's bound.

Can't comm quantum states by sending classical data.
Free entanglement is like free love
-- it changes the world.

Charles Bennett, Cambridge, 1999
Teleportation

Alice can communicate a qubit to Bob if (1) she can send 2 classical bits to Bob, and (2) they share the ebit $|\Phi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$.

How to think about quantum protocols:
Which party has what classical/quantum information?
Which party has what quantum system?
What operations he/she is allowed to do?
Teleportation

Alice can communicate a qubit to Bob if (1) she can send 2 classical bits to Bob, and (2) they share the ebit $|\Phi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$.

Schematic diagram to be completed:

$|\psi\rangle = a|0\rangle + b|1\rangle$

$|\Phi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$

- Black: Alice's
- Red: Bob's
- Blue: classical message from Alice to Bob
Main mathematical tool:
Expressing an 8-dim quantum state in 2 ways.

\[
(a|0\rangle + b|1\rangle)_{M} \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{AB}
\]

\[
= (a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle)_{MAB} \frac{1}{\sqrt{2}}
\]
Main mathematical tool:
Expressing an 8-dim quantum state in 2 ways.

\[(a|0\rangle + b|1\rangle)_{M} \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{AB}\]

\[= (a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle)_{MAB} \frac{1}{\sqrt{2}}\]

\[= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{MA} (a|0\rangle + b|1\rangle)_{B} \frac{1}{2}\]

\[+ \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)_{MA} (a|0\rangle - b|1\rangle)_{B} \frac{1}{2}\]

\[+ \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)_{MA} (a|1\rangle + b|0\rangle)_{B} \frac{1}{2}\]

\[+ \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)_{MA} (a|1\rangle - b|0\rangle)_{B} \frac{1}{2}\]
Main mathematical tool: Expressing an 8-dim quantum state in 2 ways.

\[(a|0\rangle + b|1\rangle)\_M \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)\_AB\]

\[= (a|0000\rangle + a|0111\rangle + b|1000\rangle + b|1111\rangle)\_MAB \frac{1}{\sqrt{2}} \]

\[= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)\_MA (a|0\rangle + b|1\rangle)\_B \frac{1}{2} \]

\[+ \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)\_MA (a|0\rangle - b|1\rangle)\_B \frac{1}{2} \quad \{ \text{no cross terms gives } a|0000\rangle + b|1111\rangle \}

\[+ \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)\_MA (a|1\rangle + b|0\rangle)\_B \frac{1}{2} \]

\[+ \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)\_MA (a|1\rangle - b|0\rangle)\_B \frac{1}{2} \]
Main mathematical tool:
Expressing an 8-dim quantum state in 2 ways.

\[
\begin{align*}
(a|00\rangle + b|11\rangle)_M & \quad \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_A B \\
= (a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle)_M A B & \quad \frac{1}{\sqrt{2}} \\
= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_M A (a|0\rangle + b|1\rangle)_B & \quad \frac{1}{2} \quad \text{no cross terms gives } a|000\rangle + b|111\rangle \\
+ \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)_M A (a|0\rangle - b|1\rangle)_B & \quad \frac{1}{2} \quad \text{no cross terms gives } a|011\rangle + b|100\rangle \\
+ \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)_M A (a|1\rangle + b|0\rangle)_B & \quad \frac{1}{2} \\
+ \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)_M A (a|1\rangle - b|0\rangle)_B & \quad \frac{1}{2}
\end{align*}
\]
Pauli's: $\sigma_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Bell basis:

$|\Phi_o\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$, $|\Phi_y\rangle = \frac{1}{\sqrt{2}} (|10\rangle - i|01\rangle)$

$|\Phi_x\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)$, $|\Phi_z\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)$
Pauli's: $\sigma_\varnothing = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_\chi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_\gamma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_\zeta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Bell basis:

$|\Phi_r\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$, $|\Phi_y\rangle = \frac{1}{\sqrt{2}} (\gamma |10\rangle - \gamma |01\rangle)$

$|\Phi_x\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)$, $|\Phi_z\rangle = \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle)$
Pauli's: $\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Bell basis: $|\Phi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$, $|\Phi_y\rangle = \frac{1}{\sqrt{2}} (|\Bar{1}10\rangle - i|10\rangle)$

$|\Phi_z\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)$, $|\Phi_x\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)$
If Alice measures \( MA \) along the Bell basis, each outcome \( k \in \{0, x, y, z\} \) occurs with prob 1/4, and postmeasurement state is \( |\Phi_k\rangle_{MA} \otimes 6_k |\psi\rangle_B \).
If Alice measures MA along the Bell basis, each outcome \( k \in \{0, x, y, z\} \) occurs with prob \( 1/4 \), and postmeasurement state is \( |\bar{\Psi}_k\rangle_{MA} \otimes |\psi\rangle_B \).

If Alice sends \( k \) to Bob, he can apply \( \zeta_k \) to B, turning \( \zeta_k |\psi\rangle_B \) to \( |\psi\rangle_B \).
Teleportation

Alice can communicate a qubit to Bob if (1) she can send 2 classical bits to Bob, and (2) they share the ebit $|\Phi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$.

Schematic diagram:

$$|\psi\rangle = a|00\rangle + b|11\rangle$$

$$|\Phi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$
Teleportation

Alice can communicate a qubit to Bob if (1) she can send 2 classical bits to Bob, and (2) they share the ebit $|\overline{\Phi}_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$.

Exercise: verify the following specific implementation

Here, k is given by 2 bits $(v,w)$. Note also $\delta y = 6_z \cdot 6_x$.
General: \( |\psi\rangle = \sum_\ell a_\ell |\ell\rangle |\eta_\ell\rangle \) on RS.

real ortho-normal unit vector on S

For any measurement on S given by projectors \( \{ P_k \} \)

\[ I \otimes P_k |\psi\rangle = \sum_\ell a_\ell |\ell\rangle \otimes P_k |\eta_\ell\rangle \]

\[ pr(k) = \| I \otimes P_k |\psi\rangle \|^2 = \sum_\ell a_\ell^2 \| P_k |\eta_\ell\rangle \|^2 \]

\[ = \sum_\ell a_\ell^2 \text{ tr} P_k |\eta_\ell\rangle \langle \eta_\ell | P_k \]

\[ = \sum_\ell a_\ell^2 \text{ tr} P_k |\eta_\ell\rangle \langle \eta_\ell | \]

\[ = \text{ tr} P_k \left( \sum_\ell a_\ell^2 |\eta_\ell\rangle \langle \eta_\ell | \right) \text{ where } a_\ell |\eta_\ell\rangle_S = \langle \ell | \otimes I |\psi\rangle. \]

\( \rho_S : \) density matrix on S

\[ \text{dxd} \text{d} if \ d = \text{dim}(S) \]

trace 1, positive semidefinite
Revised formulation of QM:

Revised description of quantum state:

\[ |\psi\rangle = \sum_i a_i |i\rangle |\eta_i\rangle \rightarrow |\psi\rangle \langle \psi| \rightarrow \sum_i a_i^2 |\eta_i\rangle \langle \eta_i| = \int_S \]

1. outer product 2. partial trace

revised description of measurement:

\[ \text{pr}(k) = || I \otimes P_k |\psi\rangle ||^2 \rightarrow \text{pr}(k) = \text{tr} P_k \int_S \]

Define partial trace (describing a state on S from a state on RS) so postmeasurement states & dynamics also makes sense.
The partial trace

Recall the trace of a matrix $M$ is the sum of all the diagonal elements. In the Dirac notation:

$$\text{tr} \ M = \text{tr}(M \sum_{i=1}^{d} \vert i \rangle \langle i \vert) = \sum_{i=1}^{d} \langle i \vert M \vert i \rangle$$

Definition: the partial trace of system B, denoted $\text{tr}_B$, is defined on matrices acting on systems AB as

$$\text{tr}_B \ M = \sum_{i=1}^{d} (I \otimes \langle i \vert) M (I \otimes \vert i \rangle)$$
The partial trace (example for 2 qubits)

\[ I \otimes \langle 0 | = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} [1 & 0] & [0 & 0] \\ [0 & 0] & [1 & 0] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \]

\[ I \otimes \langle 1 | = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [0 & 1] & [0 & 0] \\ [0 & 0] & [0 & 1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[(I \otimes \langle 0 |) \mathbf{M} (I \otimes |0\rangle) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{bmatrix} \]

\[(I \otimes \langle 1 |) \mathbf{M} (I \otimes |1\rangle) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_{22} & m_{24} \\ m_{42} & m_{44} \end{bmatrix} \]

\[ \dagger \text{tr}_B \mathbf{M} = \sum_{i=1}^{d} (I \otimes \langle i |) \mathbf{M} (I \otimes |i\rangle) = \begin{bmatrix} m_{11} + m_{22} & m_{13} + m_{24} \\ m_{31} + m_{42} & m_{33} + m_{44} \end{bmatrix} \]
\[ \text{tr}_B M = \begin{pmatrix} \text{tr} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} & \text{tr} \begin{pmatrix} m_{13} & m_{14} \\ m_{23} & m_{24} \end{pmatrix} \\ \text{tr} \begin{pmatrix} m_{31} & m_{32} \\ m_{41} & m_{42} \end{pmatrix} & \text{tr} \begin{pmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{pmatrix} \end{pmatrix} = M \]

Exercise:

\[ \text{tr}_A M = \text{tr}_A \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} + \begin{pmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{pmatrix} \]
Example: A, B are 3- and 2-dim respectively. \((M: 6 \times 6)\)

\[
M = \begin{pmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{pmatrix}
\]

Each \(M_{i,j}\) is a 2x2 matrix.

\[
\text{tr}_A M = M_{11} + M_{22} + M_{33}
\]  
(note, the reduced matrix on B is 2x2)

\[
\text{tr}_B M = \begin{pmatrix}
\text{tr} M_{11} & \text{tr} M_{12} & \text{tr} M_{13} \\
\text{tr} M_{21} & \text{tr} M_{22} & \text{tr} M_{23} \\
\text{tr} M_{31} & \text{tr} M_{32} & \text{tr} M_{33}
\end{pmatrix}
\]  
(note, the reduced matrix on A is 3x3)
Remark:

The trace of an r-dim system is a linear map from r x r matrices to real numbers.

The partial trace of an r-dim system is a linear map from rs x rs matrices to s x s matrices where the trace is applied to R, and the identity map on S. It acts on tensor product matrices as:

\[ \text{Tr}_R M_R \otimes M_S = (\text{Tr} M_R) \cdot M_S \]

and extends to any rs x rs matrix.
What is the most general transformation allowed by QM?

Any reasonable transformation \( N \) should take quantum states to quantum states!

Viewing \( N \) as a mapping from matrices to matrices:

1. \( N \) is linear (QM is)
2. \( N \) is trace preserving: \( \text{tr}(N(M)) = \text{tr}(M) \) (conservation of probability when \( M = \rho \))
3. \( N \) is completely positive: \( M \geq 0 \Rightarrow I \otimes N(M) \geq 0 \)

\( N \) applied to 1 out of 2 systems takes a valid initial joint state \( \rho \geq 0 \) to a valid new joint state \( I \otimes N(\rho) \geq 0 \).

E.g., hold for conjugation by unitaries and partial trace.
The identity map:
Consider the map \( \mathbb{I}(M) = M \). It is linear, trace preserving and completely positive. It represents the evolution in which nothing happens.

The identity map is most often used when one of two system is being transformed.

On a tensor product input, \( \mathbb{I} \otimes N(6 \otimes \xi) = 6 \otimes N(\xi) \).
Then, linearity allows the most general \( \mathbb{I} \otimes N(\rho) \) to be computed.
Definition: a quantum operation is a mapping from matrices to matrices that is linear, trace-preserving, and completely positive.

Synonyms: quantum channel, TCP map ...

Fairly immediate from the definition:

1. Composition of two quantum ops is a quantum op. (All 3 properties are preserved by composition.)

2. Tensor product of two quantum ops (applied to two disjoint systems) is a quantum op.
Example 1: Conjugation by unitary $N(\rho) = U \rho U^+$
Example 2: Partial trace $N(\rho) = \operatorname{tr}_R \rho_{RS}$.

Example 3: $N(\rho) = \operatorname{tr}_E (U \rho \otimes 1 \otimes 1_E U^+)$ is a quantum operation for any system $E$ and any $U$.

Proof: by examples 1-2 and composition.

Extensions: $E$ can start in any other density matrix uncorrelated with $\rho$, and partial trace can be taken over a system of any size.
Example: amplitude damping channel

We can define $U$ by its action on a pure qubit state:

$$U(a|10\rangle + b|11\rangle)_{A} = a|100\rangle_{EB} + b(\sqrt{1-\delta} |01\rangle_{EB} + \sqrt{\delta} |10\rangle_{EB})$$

the excitation is transferred from $A$ to $E$

NB $A$, $B$, $E$ all 2-dim.
Example: amplitude damping channel

We can define $U$ by its action on a pure qubit state:

$$
U (a |0\rangle + b |1\rangle) = a |00\rangle_E^B + b (\sqrt{1-\alpha} |10\rangle_E^B + \sqrt{\alpha} |11\rangle_E^B)
$$

$$
U = \begin{pmatrix}
1 & 0 \\
0 & \sqrt{1-\alpha} \\
0 & \sqrt{\alpha} \\
0 & 0
\end{pmatrix}
$$

the excitation is transfered from $A$ to $E$

NB $A$, $B$, $E$ all 2-dim.
Example: amplitude damping channel

We can define $U$ by its action on a pure qubit state:

$$U(a|0\rangle + b|1\rangle) = a|0\rangle_E + b\left(\sqrt{1-\delta} |0\rangle + \sqrt{\delta} |1\rangle\right)_E$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\delta} \\ 0 & \sqrt{\delta} \\ 0 & 0 \end{pmatrix}$$

the excitation is transferred from $A$ to $E$

NB $A$, $B$, $E$ all 2-dim.

On a general density matrix $\rho = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$,

$$U \rho U^+ = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\delta} \\ 0 & \sqrt{\delta} \\ 0 & 0 \end{pmatrix} \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\delta} & 0 & 0 \\ 0 & 0 & \sqrt{\delta} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{bmatrix} c \sqrt{1-\delta} & \sqrt{\delta}d & 0 \\ \sqrt{1-\delta}e & (1-\delta)f & \sqrt{\delta}f \\ \sqrt{\delta}e & \sqrt{\delta}f & (1-\delta)f \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
\[ tr_{E} U \circ U^{+} = tr_{E} \left( \begin{array}{cccc}
    c & \sqrt{f_{d}} & s & 0 \\
    \sqrt{f_{e}} e^{(H_{e})f} & \sqrt{f_{d}} f & 0 \\
    \sqrt{f_{e}} s & \sqrt{f_{d}} f & 0 \\
    0 & 0 & 0 & 0 \\
\end{array} \right) \]

\[ = \left( \begin{array}{c}
    c \sqrt{f_{d}} \\
    \sqrt{f_{e}} e^{(H_{e})f} \\
\end{array} \right) + \left( \begin{array}{cc}
    0 & 0 \\
\end{array} \right) \]

\[ = \left( \begin{array}{c}
    c + rf \sqrt{f_{d}} \\
    \sqrt{f_{e}} e^{(H_{e})f} \\
\end{array} \right) \]
\[ \text{tr}_E \ U \rho \ U^+ = \text{tr}_E \begin{pmatrix} c & \sqrt{rf}d & \sqrt{rf}d & 0 \\ \sqrt{rf}e & (rf)f & \sqrt{rf}f & 0 \\ \sqrt{rf}e & \sqrt{rf}f & rf & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ = \begin{pmatrix} c & \sqrt{rf}d \\ \sqrt{rf}e & (rf)f \end{pmatrix} + \begin{pmatrix} rf & 0 \\ 0 & 0 \end{pmatrix} \]

\[ = \begin{pmatrix} c + rf & \sqrt{rf}d \\ \sqrt{rf}e & (rf)f \end{pmatrix} \]

So, the channel takes \( \rho = \begin{pmatrix} c & d \\ e & f \end{pmatrix} \) to \( \begin{pmatrix} c + rf & \sqrt{rf}d \\ \sqrt{rf}e & (rf)f \end{pmatrix} \)

A fraction \( \chi \) of the (1,1) entry is moved to the (0,0) entry, and the off diagonal terms are diminished.
What is $N(\rho)$ in terms of $U$?

Let

$$U = \sum_{j=0}^{d_E-1} \sum_{k=0}^{d_E-1} |j \times k|_E \otimes U_{jk} = \begin{pmatrix} U_{00} & U_{01} & U_{02} & \cdots \\ U_{10} & U_{11} & U_{12} & \cdots \\ U_{20} & U_{21} & U_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

E: 1st register.
What is \( N(\rho) \) in terms of \( U \)?

Let

\[
U = \sum_{j=0}^{d_{E-1}} \sum_{k=0}^{d_{E-1}} |j\rangle \langle k|_E \otimes U_{jk} = \begin{pmatrix}
U_{00} & U_{01} & U_{02} \\
U_{10} & U_{11} & U_{12} \\
U_{20} & U_{21} & U_{22} \\
\vdots & \vdots & \vdots
\end{pmatrix}
\]

\( E \): 1st register.

\[
N(\rho) = \text{tr}_E \left( U \rho \otimes 1_0 \otimes 1_E \right) U^+
\]

\[
= \text{tr}_E \left( \sum_{j=0}^{d_{E-1}} \sum_{k=0}^{d_{E-1}} |j\rangle \langle k|_E \otimes U_{jk} \right) \left( |0\rangle \langle 0|_E \otimes \rho \right) \left( \sum_{j'=0}^{d_{E-1}} \sum_{k'=0}^{d_{E-1}} |k'\rangle \langle j'|_E \otimes U^*_{j'k'} \right)
\]
What is $N(\rho)$ in terms of $U$?

Let

$$U = \sum_{j=0}^{d_{E-1}} \sum_{k=0}^{d_{E-1}} |j \rangle \langle k|_E \otimes U_{jk}$$

$E$: 1st register.

$$N(\rho) = \text{Tr}_E \left( U \rho \otimes 1 \otimes 0 \otimes 1 \otimes U^* \right)$$

$$= \text{Tr}_E \left( \sum_{j=0}^{d_{E-1}} \sum_{k=0}^{d_{E-1}} |j \rangle \langle k|_E \otimes U_{jk} \right) \left( |0 \rangle \langle 0|_E \otimes \rho \right) \left( \sum_{j'=0}^{d_{E-1}} \sum_{k'=0}^{d_{E-1}} |k' \rangle \langle j'|_E \otimes U^*_{j'k'} \right)$$

$$= \text{Tr}_E \left( \sum_{j=0}^{d_{E-1}} |j \rangle \langle j|_E \otimes U_{j0} \right) (1 \otimes \rho) \left( \sum_{j=0}^{d_{E-1}} \langle j'|_E \otimes U^*_{j'0} \right)$$

Isometry

1-dim

Can be omitted.
What is $N(\rho)$ in terms of $U$?

Let $U = \sum_{j=0}^{d_{E-1}} \sum_{k=0}^{d_{E-1}} 1_j X 1_k |E \times \otimes U_{jk} = E: 1\text{st register.}$

$N(\rho) = \text{tr}_E (U \rho \otimes 1 E \otimes U^+ )$

$= \text{tr}_E \left( \sum_{j=0}^{d_{E-1}} \sum_{k=0}^{d_{E-1}} 1_j X 1_k |E \otimes U_{jk} \right) \left( 1 \times 0 |E \otimes \rho \right) \left( \sum_{j'=0}^{d_{E-1}} \sum_{k'=0}^{d_{E-1}} 1_{j'} X 1_j |E \otimes U_{j'k'}^+ \right)$

$= \text{tr}_E \left( \sum_{j=0}^{d_{E-1}} 1_j |E \otimes U_{j0} \right) \left( 1 \otimes \rho \right) \left( \sum_{j=0}^{d_{E-1}} \langle j' |E \otimes U_{j0}^+ \right) \text{ isometry}$

$= \sum_{j=0}^{d_{E-1}} U_{j0} \rho U_{j0}^+ \text{ mixture of states } \frac{U_{j0} \rho U_{j0}^+}{\text{tr} U_{j0}^+ U_{j0} \rho}$

not nec unitary

\[ \text{isometry} \]

\[ \begin{pmatrix}
U_{00} & U_{01} & U_{02} & \cdots \\
U_{10} & U_{11} & U_{12} & \cdots \\
U_{20} & U_{21} & U_{22} & \cdots \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix} \]
More generally, let $U$ be an isometry taking system $A$ to system $BE$ (dims of $A$, $B$, and $E$ are arbitrary).

$$U = \begin{pmatrix} \bar{A}_1 & \bar{A}_2 & \cdots & \bar{A}_K \\ \bar{A}_K & \bar{A}_K & \cdots & \bar{A}_K \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{d_E} & \bar{A}_{d_E} & \cdots & \bar{A}_{d_E} \end{pmatrix}$$

$d_E$ blocks each taking $d_A$ to $d_B$ dims

$$U = \sum_{k=1}^{d_E} |k\rangle_E \otimes A_k$$

Stinespring dilation, isometric extension
More generally, let $U$ be an isometry taking system $A$ to system $BE$ (dims of $A$, $B$, and $E$ are arbitrary).

$$U = \uparrow \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_k \\ \vdots \\ A_{d_E} \end{array} \downarrow \begin{array}{c} d_B \\ d_B \\ \vdots \\ d_B \end{array}$$

$$N(\rho) = \operatorname{Tr}_E \left( U \rho U^+ \right) = \sum_{k=1}^{d_E} A_k \rho A_k^+$$

Kraus representation of $N$ $A_k$'s: Kraus operators

not $A_k^+$'s

Stinespring dilation, isometric extension

$$U = \sum_{k=1}^{d_E} |k\rangle_E \otimes A_k$$
More generally, let $U$ be an isometry taking system $A$ to system $BE$ (dims of $A$, $B$, and $E$ are arbitrary).

$$U = \begin{array}{c|c|c}
A_1 & & d_B \\
\hline
A_2 & & d_B \\
\hline
A_K & d_E & d_E \\
\hline
A_{d_E} & &
\end{array}$$

$$d_B d_E$$

Each taking $d_A$ to $d_B$ dims

$$d_E$$ blocks

$$N(\rho) = tr_E( \rho U U^\dagger ) = \sum_{k=1}^{d_E} A_K \rho A_K^\dagger$$

Kraus representation of $N$

$A_K$'s: Kraus operators

* A map w/ Kraus representation is linear and completely positive

$$U = \sum_{k=1}^{d_E} |k\rangle_E \otimes A_K$$

Stinespring dilation, isometric extension
More generally, let $U$ be an isometry taking system $A$ to system $BE$ (dims of $A$, $B$, and $E$ are arbitrary).

$$U = \begin{array} \end{array}$$

$$U = \sum_{k=1}^{d_E} |k\rangle_E \otimes A_k$$

Stinespring dilation, isometric extension

$$N(\rho) = \text{tr}_E (U \rho U^+) = \sum_{k=1}^{d_E} A_k \rho A_k^+$$

Kraus representation of $N$

$A_k$'s: Kraus operators

* A map w/ Kraus representation is linear and completely positive

* $U$ isometry $\iff U^+U = I_A$

$\iff \sum_{k=1}^{d_E} A_k^+ A_k = I_A$

$\iff N$ trace preserving
Example: amplitude damping channel

\[ U = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-x} \\ 0 & \sqrt{x} \\ 0 & 0 \end{pmatrix} \qquad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-y} \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & \sqrt{x} \\ 0 & 0 \end{pmatrix} \]

\[ N(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger \]

Ex: check \( A_0^\dagger A_0 + A_1^\dagger A_1 = I \)
Example: amplitude damping channel

\[
U = \begin{pmatrix}
1 & 0 \\
0 & \sqrt{1-\gamma} \\
0 & \sqrt{\gamma} \\
0 & 0
\end{pmatrix}
\quad A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}
\quad A_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}
\]

\[N(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger\]

Ex: check \[A_0^\dagger A_0 + A_1^\dagger A_1 = I\]

If the initial state is \[|\psi\rangle = a |0\rangle + b |1\rangle \quad (\rho = |\psi\rangle \langle \psi|)\]
output is the mixture of two unnormalized states:

\[A_0 |\psi\rangle = a |0\rangle + \sqrt{1-\gamma} b |1\rangle\]
\[A_1 |\psi\rangle = \sqrt{\gamma} b |1\rangle\]
Example: amplitude damping channel

\[ U = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \\ 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \]

\[ A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \]

\[ N(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger \]

Ex: check \( A_0^\dagger A_0 + A_1^\dagger A_1 = I \)

If the initial state is \( |\psi\rangle = a |0\rangle + b |1\rangle \) (\( \rho = \langle \psi | \psi \rangle \)) output is the mixture of two unnormalized states:

\[ A_0 |\psi\rangle = a |0\rangle + \sqrt{1-\gamma} b |1\rangle \]

\[ A_1 |\psi\rangle = \sqrt{\gamma} b |1\rangle \]

Interpretation: \( |0\rangle \) : ground state

\( |1\rangle \) : excited state

\( A_1 \) : de-excitation (with prob \( \gamma \))

\( A_0 \) : no de-excitation, but diminished amplitude for \( |1\rangle \)
Exercise: evaluate $N\left(\frac{1}{2}I + aX + bY + cZ\right)$ and find how $N$ transform the Bloch sphere.

The ground state $|\psi\rangle\langle\psi|$ is a fixed point of $N$. $N$ is not unital (taking the identity matrix to itself).
**Theorem:** any quantum operation $N$ from system $A$ to system $B$ can be represented as $N(\rho) = \text{tr}_E (U \rho U^\dagger)$ for some system $E$ and some Stinespring dilation $U$.

Proof omitted. See arxiv.org/abs/quant-ph/0201119
Representations of quantum operations:

1. Unitary representation

\[ N(\rho) = \text{Tr}_E ( U \rho U^+) \]

\[ \rho \xrightarrow{A} U \xrightarrow{B} N(\rho) \]

\[ \leftarrow \text{partial trace} \]

2a. Kraus rep: \[ N(\rho) = \sum_{k=1}^{d_E} A_k \rho A_k^+ \], \[ \sum_{k=1}^{d_E} A_k^+ A_k = I_A \]

2b. Conversely, given \( d_E \) operators \( A_k \) mapping from system A to B satisfying

\[ \sum_{k=1}^{d_E} A_k^+ A_k = I_A \]

\[ U = \sum_{k=1}^{d_E} |k\rangle_E \otimes A_k \]

is an isometry, and \( \text{Tr}_E ( U \rho U^+) = \sum_{k=1}^{d_E} A_k \rho A_k^+ \)

3. \( N(\rho) \) as an explicit function of \( \rho \) e.g. \[
\begin{pmatrix}
  c & d \\
  e & f
\end{pmatrix} \rightarrow
\begin{pmatrix}
  c+if & \text{Tr}(\rho) \cdot d \\
  \text{Tr}(\rho) \cdot e & \text{Tr}(\rho) \cdot f
\end{pmatrix}
\]

4. Choi matrix (reading)