Lec 21.

Last time: $Q_{\to}(N) = Q(N) \quad \forall N$.

To be continued: EPP-1 & $Q_{\to}$. (later).

What about $Q_{\leftarrow}$ & $Q_{\leftrightarrow}$?

<u>Thm:</u> $Q_{\leftrightarrow}(N) = E_{\leftrightarrow}(N)$ (ent capacity of N given free 2-way cc).

(Error : measured as $\| \rho_{out} - \Phi^{\otimes nr} \|_1$)

<u>Pf idea:</u> " $\geq$ "

If Alice & Bob can create $n E_{\leftrightarrow}(N)$ ebits with error $\leq \varepsilon_n$ (in trace distance) using N n times & 2-way cc, then Alice can teleport $n E_{\leftrightarrow}(N)$ qubits using these ebits and free forward cc, with same error in trace distance.

$$\left[\begin{array}{l} nN + 2nE_{\leftrightarrow} \text{ cbits}_{\rightarrow} + n_1 \text{ cbits}_{\rightarrow} \\ \qquad\qquad\qquad\qquad\qquad + n_2 \text{ cbits}_{\leftarrow} \\ \geq nE_{\leftrightarrow} \text{ ebits} + 2nE_{\leftrightarrow} \text{ cbits}. \\ \geq nE_{\leftrightarrow} \text{ qbits.} \end{array}\right]$$

" $\leq$ " If Alice can send $n Q_{\leftrightarrow}(N)$ qubits to Bob using N n times & free 2-way cc with error $\leq \varepsilon_n$ then she can send halves of $n Q_{\leftrightarrow}(N)$ ebits to Bob with the same error.

$$\left[\begin{array}{l} nN + n_1 \text{ cbits}_{\rightarrow} + n_2 \text{ cbits}_{\leftarrow} \\ \geq nQ_{\leftrightarrow} \text{ qbits} \\ \geq nQ_{\leftrightarrow} \text{ ebits.} \end{array}\right]$$

eg. If N = erasure channel w/ prob error p.

then $Q_{\leftrightarrow}(N) = 1-p$.

(cf $Q(N) = 1-2p$, $C(N) = 1-p$, $C_E(N) = 2(1-p)$).

How? fix ε, δ

① Alice sends n halves of n ebits using $N^{\otimes n}$.

② Bob tells Alice which are erased

③ They discard the bad pairs.

Prob (fewer than $n(1-p-\delta)$ good ebits) $\underbrace{\leq \exp(-n\delta^2 \cdot \text{const})}_{\text{typical large deviation bdd}}$

④ Alice teleports $n(1-p-\delta)$ qubits to Bob.

rate $\to n(1-p)$. Error $= 0$.

Clearly can't do better.

For most other channels, neither $Q_{\leftrightarrow}$ nor $E_{\leftrightarrow}$ has been understood much.

- How about $Q_{\leftarrow}(N)$? O/c: $Q(N) \leq Q_{\leftarrow}(N) \leq Q_{\leftrightarrow}(N)$

- no expression

- no link to $E_{\leftarrow}$

- known: $\exists N$ s.t. $Q(N) \underset{\neq}{<} Q_{\leftarrow}(N)$. ⓐ

- known: $\exists N$ s.t. $Q_{\leftarrow}(N) \underset{\neq}{<} Q_{\leftrightarrow}(N)$ ⓑ.

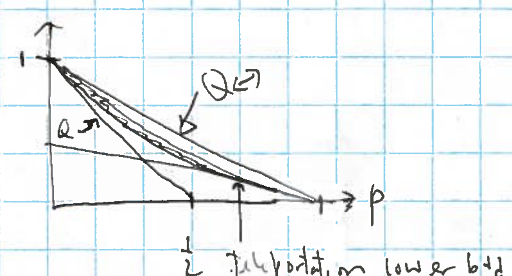
To see ⓐ, consider Erasure channel with erasure prob $p \geq \frac{1}{2}$.

We know $Q(N)=0$

But $Q_{\leftarrow}(N) \geq \frac{1}{3}(1-p)$

same teleportation method as in $Q_{\leftrightarrow}$
but forward cc is not free, but
supplied by N itself.

ⓑ much harder to show .... relies on sender not
knowing whether Bob or Eve will receive each transmission
so limit how much she can favor Bob over Eve.



- Note: if N classical,

$$C(N) = C_{\leftarrow}(N)$$

ie feedback does not increase classical capacity of classical channel.

Reason: converse holds even with feedback.

← though it can improve code performance or finite block length rate-distortion trade off.

- Likewise, $C_E(N) = C_{E,\leftarrow}(N)$.

- Quantum feedback (free Q channel from Bob to Alice) increases $C(N)$ to $C_E(N)$ & $Q(N)$ to $Q_E(N)$ (by making entanglement free) but no more (Gary Bowen)

7-8 pages.

$Q_B = Q_\leftarrow$

$Q_2 = Q_\leftrightarrow$

Lec 21, July 20, 2010.

Note Title 20/07/2010

For any 2 capacities

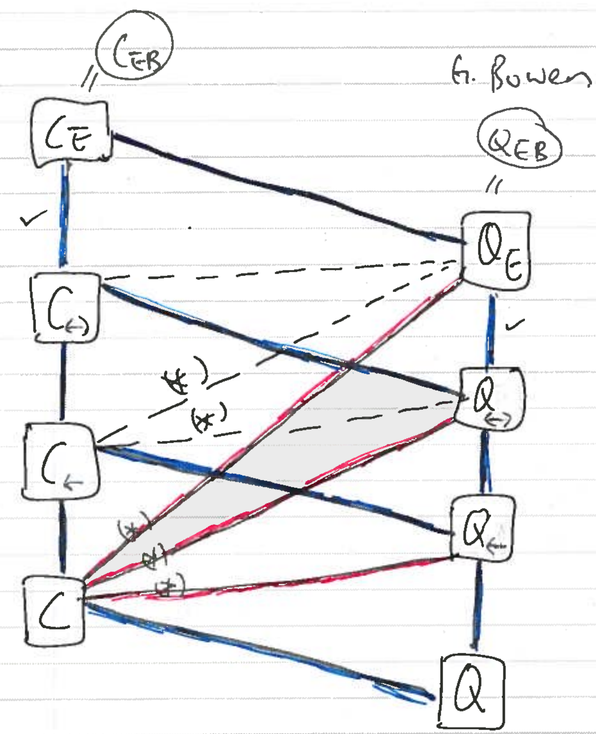
are they related by "$\leq$" $\forall N$

or are they incomparable,

or we don't know?

0406086 Bennett, Devetak, Shor, Smolin

0904.4050 Smith, Smolin.



(*) R_2 — C small but $Q_\leftarrow$ $Q_\leftrightarrow$ Q_E all large.

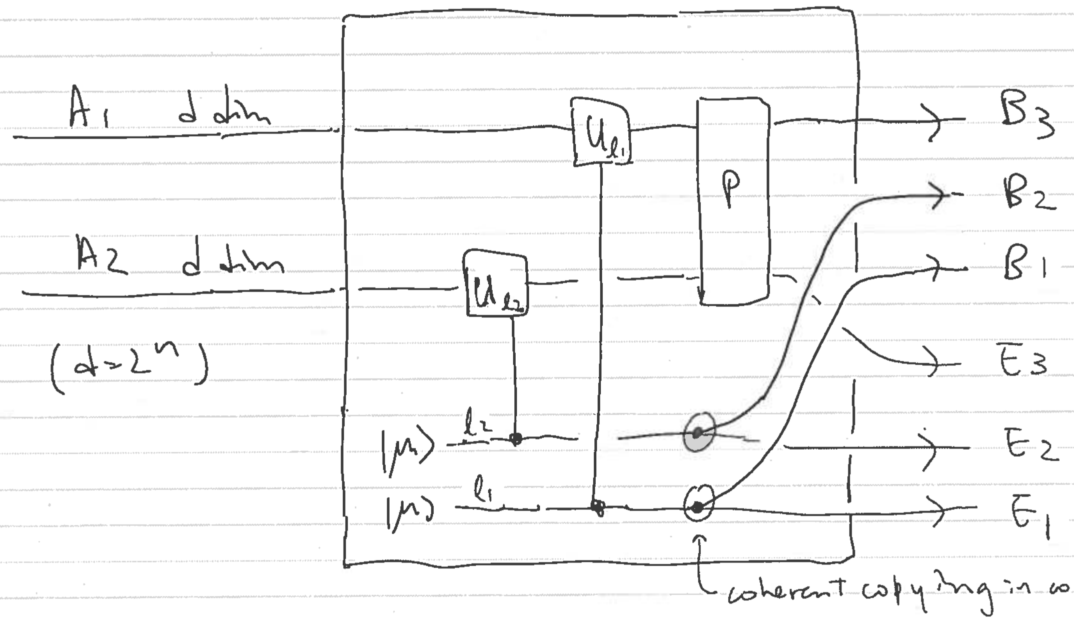
$\underbrace{\text{classical channel.}}$ (*)

$C_E = 1$ $\therefore Q_E = \frac{1}{2}$

But $C = 1$, $Q \leq Q_\leftarrow \leq Q_\leftrightarrow \leq Q_E = \frac{1}{2}$

★ Place in middle left.

The Rocket channel R_d (iso ext):



(NB Neither degradable (anti deg).

(a new (l_1, l_2) is drawn for every use of R_d)

$$|\mu\rangle = \frac{1}{\sqrt{|C_n|}} \sum_{l=1}^{|C_n|} |l\rangle, \quad P|i\rangle_{A_1}|j\rangle_{A_2} = \omega^{ij}|i\rangle_{A_1}|j\rangle_{A_2}$$

C_n : Clifford group on n qubits, ω = primitive dth root of unity

Properties of R_d:

1. $C(R_d) \leq 2 \quad \therefore \quad \underbrace{Q(R_d) \leq P(R_d) \leq C(R_d)}_{\text{general ineq } \forall \text{ channel}} \leq 2$

2. $Q_E(R_d) \geq \log d$, SD coding $\Rightarrow \begin{cases} P_E(R_d) \geq 2\log d \\ C_E(R_d) \geq 2\log d \end{cases}$

3. $Q^{(1)}(R_d \otimes E_{\frac{1}{2}}) \geq \frac{1}{2}\log d \quad \therefore Q_{ss}(R_d) \geq \frac{1}{2}\log d$

4. $Q_{\leftrightarrow}(R_d) \geq \log d$

   $Q_{\leftarrow}(R_d) \geq \frac{1}{2}\log d$

   $C_{\leftarrow}(R_d) \geq \frac{2}{3}\log d$

① The channel is designed to suppress Q & C :

- U_{ℓ_2} unknown to sender

  Else Sender can input $U_{\ell_2}^{-1}|0\rangle$ to A_2

  and $A_1 \to B_3$ will then be a noiseless Q channel, $Q(R_d) \geq \log d$

- U_{ℓ_1} unknown to sender

  Else $U_{\ell_1}^{-1}|i\rangle$ for $i = 1 \ldots d$ will emerge as phase $\times\, |i\rangle$

  and $A_1 \to B_3$ will be a noiseless C classical, $C(R_d) \geq \log d$

- See 0904.4050 for a proof why $C \leq 2$.

  Main idea: $\forall$ input, averaged over (ℓ_1, ℓ_2), $B_3^{\otimes n}$ & $E_3^{\otimes n}$ very entangled

  This can be rigorously proved: $\forall \rho$ $\quad S(R_d^{\otimes n}(\rho)) \geq n(\log d - 2)$

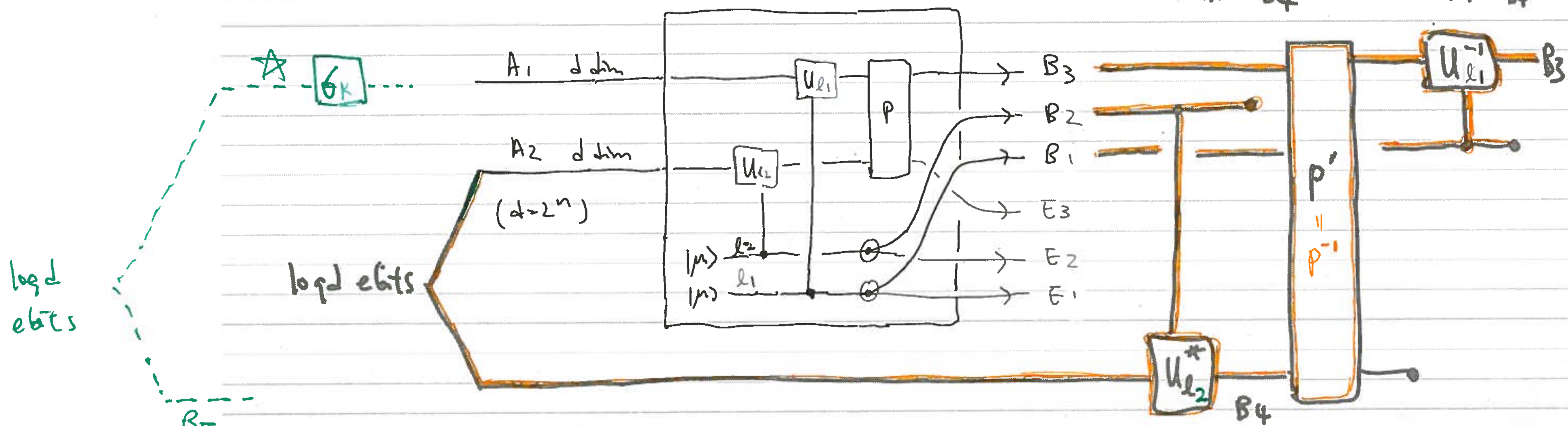
  So an upper bound $C(R_d) \leq 2$ (even if $\chi^{(1)} \neq C$).

② $Q_E(R_d) \geq \log d$:

R_d is made to be "echo-correctable" or "retro correctable"

If A & B can consume $\log d$ ebits

$$P'|i\rangle_{A_1}|j\rangle_{B_4} = \omega^{-ij}|i\rangle_{A_1}|j\rangle_{B_4}$$



Then $A_1 \to \square \to B_3$ is identity channel on d dims.

$\therefore Q_E(R_d) \geq \log d$,

In fact: R_d + log d ebits $\geq$ log d qbits (2a). ← keep on board

1-shot, exact

☆ Compose w/ SP:

$\boxed{R_d + 2\log d \text{ ebits}} \geq \log d$ qbits + $\log d$ ebits

$\geq$ 2 log d cobits $\geq$ 2 log d P-bits or 2 log d c bits.

(2b)

$\therefore C_E, P_E \geq 2\log d$

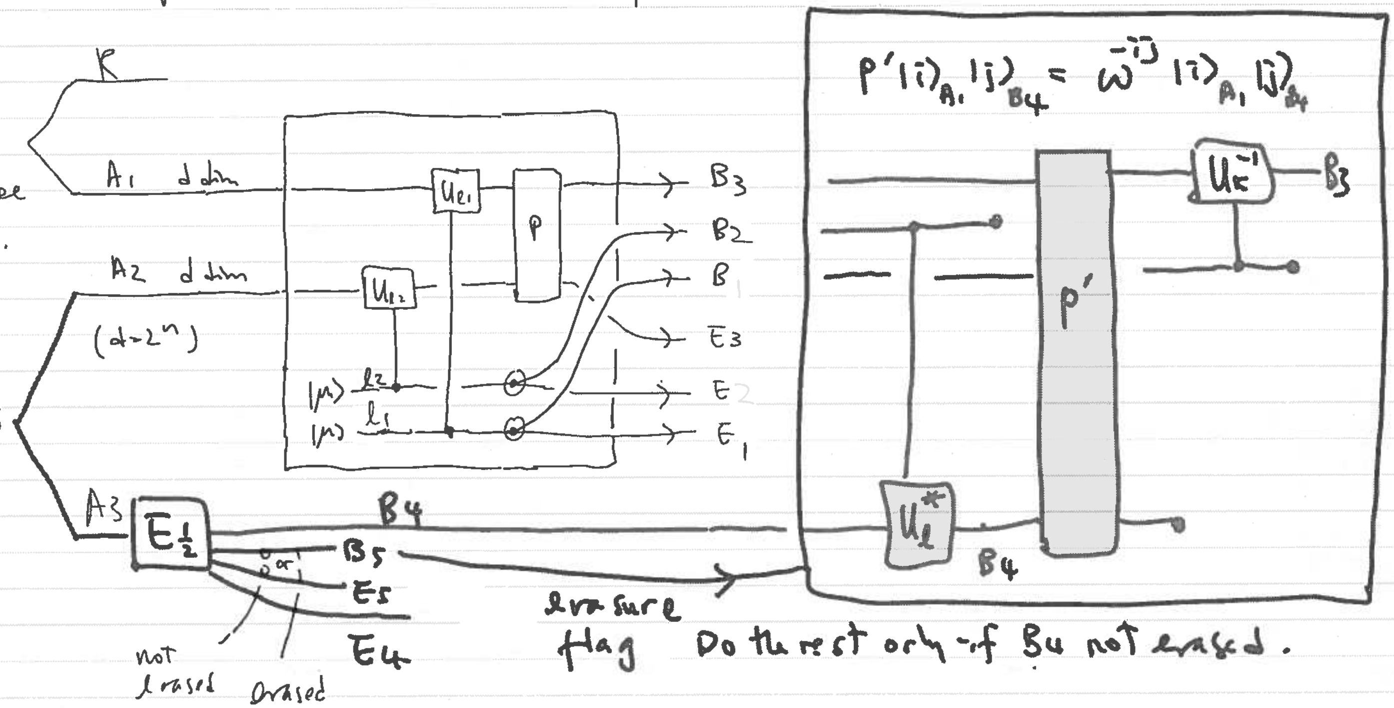
③ $Q^{(1)}(R_d \otimes E_{\frac{1}{2}}) \geq \frac{1}{2} \log d$.

Idea: use $E_{\frac{1}{2}}$ to general $\log d$ ebits wp $\frac{1}{2}$

Then proceed w/ entanglement assisted communication.

To bound $Q^{(1)}(R_d)$

Let RA_1 be max ent.



Recall :

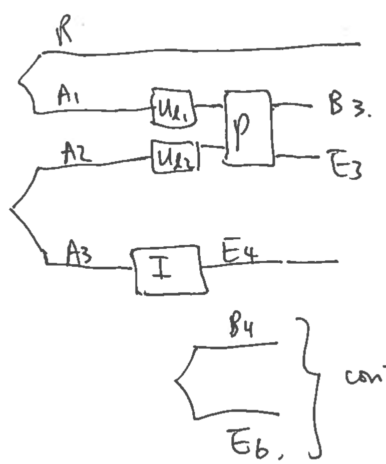
If $\rho_{RB} = \sum_i p_i \rho^{(i)}_{R\tilde{B}} \otimes |i\rangle\langle i|_C$, $B = \tilde{B}C$

then $I_c(R\rangle B)_\rho = \sum_i p_i \, I_c(R\rangle \tilde{B})_{\rho^{(i)}_{R\tilde{B}}}$

Apply to the state $R\, B_3\, B_1\, B_2\, B_4\, B_5$. $[C = B_5\, B_1\, B_2]$

- wp $\frac{1}{2}$, B_5 is in state $|0\rangle\langle 0|$, correction done
  then $\forall$ values in $B_1 B_2$
  we have $|\Phi_d\rangle$ in $R\, B_3$ in the end.

- wp $\frac{1}{2}$, B_5 is in state $|1\rangle\langle 1|$.
  "correction" is skipped.
  Further condition on $B_1 B_2$ being $|e_1\rangle\langle e_1| \otimes |e_2\rangle\langle e_2|$ (same for $E_1 E_2$)



contributes 0 to I_c

U_{e_1}, U_{e_2} do not affect I_c.

$S_B = \log d$
$S_E = \log d.$

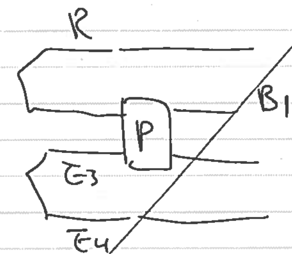
$\left.\begin{matrix} \\ \end{matrix}\right\}$ P = condition on $B_3 = |i\rangle\langle i|$, apply Z^i to E_3

$I_c = 0$.

wp $\frac{1}{2}$, B_4 not erased. $I_c(R \rangle B_1) = \log d$

wp $\frac{1}{2}$, B_4 erased, • A_2 & E_4 max ent

• for each l, k (known to Bob & Eve)



and $S(RB_1) = S(B_1) = \log d$

$\therefore I_c(R \rangle B_1) = \log d$

(in both cases, other sys with Bob irrelevant in calculating I_c.)

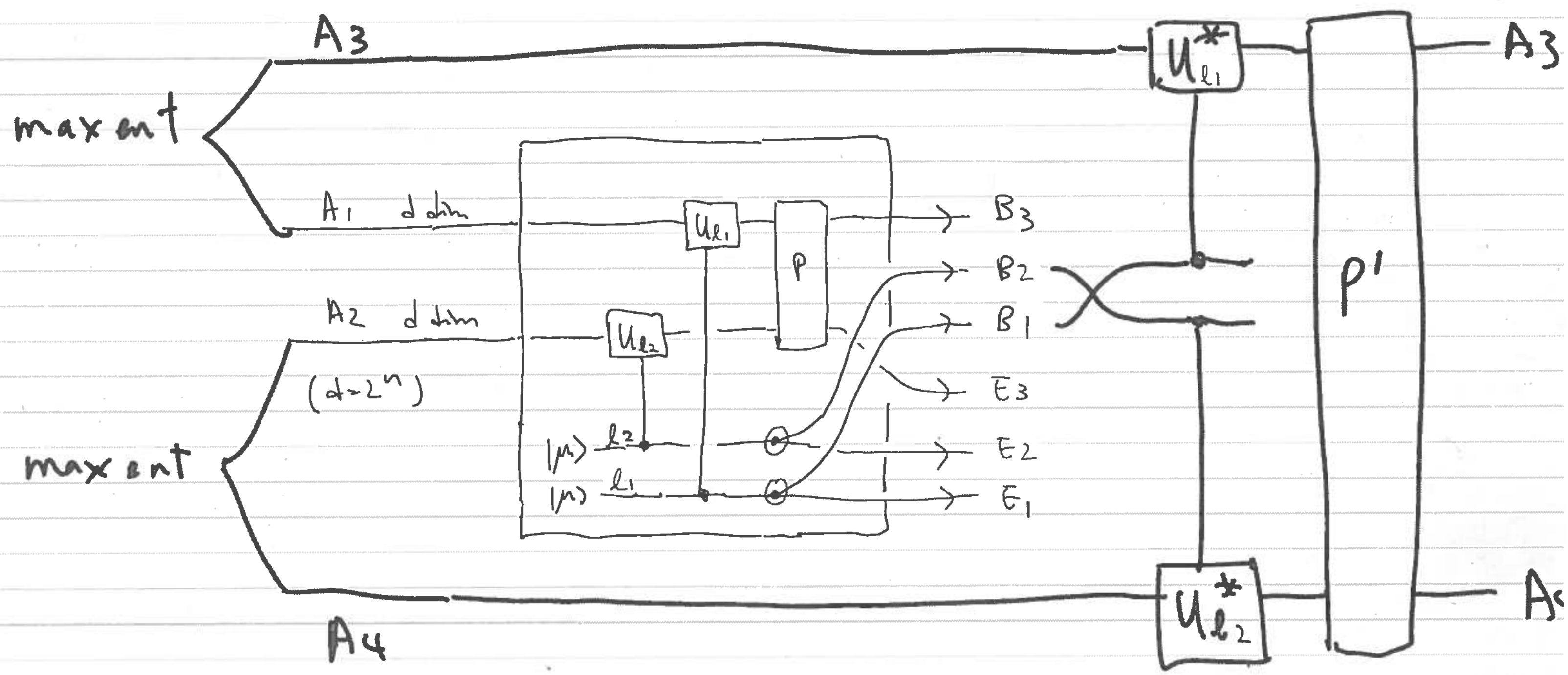
$\therefore Q^{(1)}(R_d \otimes E_{\frac{1}{2}}) \geq \frac{1}{2}\log d + \frac{1}{2} \cdot 0 = \frac{1}{2}\log d$

(Though $Q(R_d) \leq 2$, $Q(E_{\frac{1}{2}}) = 0$, very superadditive.)

$\therefore Q_{ss}(R_d) \geq \frac{1}{2}\log d$.

④ $Q_{\leftrightarrow}(R_d) \geq \log d$.

Give free 2-way comm, Bob meas $B_1 B_2$ and send them to Alice. For simplicity, Bob sends $B_1 B_2$ to Alice.



So $A_3 B_3$ max ent (& $E_3 A_4$ max ent).

$(R_d + 2\log|C_n|\ \text{cbit}_{\leftarrow} \geq \log d\ \text{ebits}) \sim$ 4a

or $E_{\leftrightarrow}(R_d) \geq \log d$.

Alice uses free forward CC to teleport log d qubits

$(R_d + 2\log|C_n|\ \text{cbit}_{\leftarrow} + 2\log d\ \text{cbit}_{\rightarrow} \geq \log d\ \text{qbit}_{\rightarrow})$ 4b

$\therefore Q_{\leftrightarrow}(R_d) \geq \log d$.

& $Q_{\leftrightarrow}(R_d) = E_{\leftrightarrow}(R_d) \geq \log d$.

If only free backward CC is available, we concatenate 4a & 2a ie generate log d ebits & run ent assisted protocol

$$2R_d + 2\log|C_n|\ \text{cbit}_{\leftarrow} \underset{4a}{\geq} R_d + \log d\ \text{ebit} \underset{2a}{\geq} \log d\ \text{qbit}$$

$\therefore Q_{\leftarrow} \geq \frac{1}{2}\log d$.

ie create ent w/ back CC, then use $Q_{=}$ protocol + SD coding.

Finally, using 4a twice & 2b:

Ex: draw the circuit diagram w/ $3R_d$ generating $2\log d$ cbits!

$$3R_d + 4\log|C_n|\ \text{cbit}_{\leftarrow} \underset{4a \times 2}{\geq} R_d + 2\log d\ \text{ebit}$$

$$\overset{2b}{\geq} 2\log d\ \text{cbit}.$$

$\therefore C_{\leftarrow}(R_d) \geq P_{\leftarrow}(R_d) \geq \frac{2}{3}\log d.$

$C_{\leftrightarrow}(R_d) \geq \log d$ from $Q_{\leftrightarrow}(R_d) \geq \log d$ but ~~I don't know how to make $C_{\leftrightarrow}$ larger~~.

~~it~~ far from upper bdd $C_E = 2\log d$.