Lec 20

Last time:

BSST Thm: $C_E(N) = \max_{|\psi\rangle_{RA}} S(R:B)_{I \otimes N(|\psi\rangle\langle\psi|)}$

eg. 1 If $N = I$, $|\psi\rangle_{RA} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ optimal

$C_E(N) = 2$ superdense coding

eg2. If N_p = erasure channel w/ error p.

$$I \otimes N_p(|\psi\rangle\langle\psi|_{RA}) = (1-p)|\psi\rangle\langle\psi|_{RB} + p\left(\mathrm{tr}_B |\psi\rangle\langle\psi|\right)_R \otimes |2\rangle\langle 2|_B$$

$$S(R:B)_{I\otimes N_p(|\psi\rangle\langle\psi|_{RA})}$$

$$= S(R) + S(B) - S(RB)$$

$$= S(\mathrm{tr}_B |\psi\rangle\langle\psi|) + [H(p) + (1-p)S(\mathrm{tr}_R|\psi\rangle\langle\psi|) + p\cdot 0]$$
$$- [H(p) + (1-p)\cdot 0 + p\,S(\mathrm{tr}_B|\psi\rangle\langle\psi|)]$$

$$= (1+1-p-p)\,S(\mathrm{tr}_B|\psi\rangle\langle\psi|)$$

$$= 2(1-p)\,S(\mathrm{tr}_B|\psi\rangle\langle\psi|)$$

$$\Rightarrow C_E(N_p) = \max_{|\psi\rangle_{RA}} 2(1-p)\,S(\mathrm{tr}_B|\psi\rangle\langle\psi|)$$

$$= 2(1-p) \quad (\text{when } |\psi\rangle_{RA} = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)).$$

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eg3. N classical, then $C_E(N) = C(N)$. (Ex.)

eg4. N_q = depolarizing channel. As $q \to 1$, $C(N), C_E(N) \to 0$
$N_q(\rho) = (1-q)\rho + q\frac{I}{2}$. but $\frac{C_E(N)}{C(N)} \to \infty$.

eg. N classical, then $C_E(N) = C(N)$. (Entanglement doesn't help ♀ capacity.)

Pf. Let $\{|x\rangle\}$, $\{|y\rangle\}$ be special bases

corresponding to the classical inputs/outputs,

N be specified by $p(y|x)$.

i.e. $N(\rho) = \sum_y \sum_x \mathrm{tr}(\rho\, |x\rangle\langle x|)\; p(y|x)\; |y\rangle\langle y|$

① $I \otimes N(|\psi\rangle\langle\psi|)$ for general $|\psi\rangle_{RA}$:

Let $|\psi\rangle = \sum_x \sqrt{p_x}\; |\phi_x\rangle_R |x\rangle_A$

normalized but not nec ortho.

$\left[\text{If } |\psi\rangle = \sum_{ax} c_{ax} \underbrace{|a\rangle_R |x\rangle_A}_{\text{bases}}, \text{ then } |\phi_x\rangle = \frac{\sum_a c_{ax} |a\rangle}{\sqrt{\sum_a |c_{ax}|^2}}. \right]$

$(I \otimes \langle x|)\, |\psi\rangle = \sqrt{p_x}\; |\phi_x\rangle_R$

$$I \otimes N \underbrace{(|\psi\rangle\langle\psi|)}_{RA} = \sum_y \sum_x p_x\, |\phi_x\rangle\langle\phi_x|_R \otimes p(y|x)\; |y\rangle\langle y|_B . \qquad ①$$

Evaluate $S(R:B)$ & max over $|\psi\rangle$ gives $C_E(N)$.

② Claim: using input distribution p_x & Shannon's (unassisted) code achieves $I(X:Y) \geq$ ①.

Pf: Let $\rho = \sum_x p_x\, |x\rangle\langle x|_R \otimes |x\rangle\langle x|_A$

$$\underbrace{I \otimes N(\rho) = \sum_x p_x\, |x\rangle\langle x|_R \otimes p(y|x)\; |y\rangle\langle y|_B}_{S(R:B) \text{ eval on this } = I(X:Y)} \qquad ②.$$

From ②, apply local op to create $|\Phi_x\rangle$ from $|x\rangle$ followed by discarding gives ①.

$$\therefore\ S(R:B)_{②} \geq S(R:B)_{①}.$$

$$\therefore\ \underbrace{\max_{p(x)} I(X:Y)}_{C(N).} \geq \max_{|\psi\rangle} S(R:B)_{I\otimes N(|\psi\rangle\langle\psi|)} = C_E(N).$$

From ①-③, tempting to conjecture:

- $\frac{C_E(N)}{C(N)} \leq 2 \quad \forall (N)$
- $C_E(N) = C(N)$ for entanglement breaking channels

Both false.

④ Let $N(\rho) = (1-q)\rho + q\frac{I}{d}$ (d-dim dep channel)

- Then $C(N) = \log d - H_d\left(1 - x\left(\frac{d-1}{d}\right)\right)$

entropy of d-dary distⁿ
with 1 element of prob p,
d-1 elements each of prob $\frac{1-p}{d-1}$.

(Achieved on input ensemble $|0\rangle, |1\rangle, \ldots, |d-1\rangle$ with uniform prob.)

- And $C_E(N) = 2\log d - H_{d^2}\left(1 - x\left(\frac{d^2-1}{d^2}\right)\right)$. ← ☆

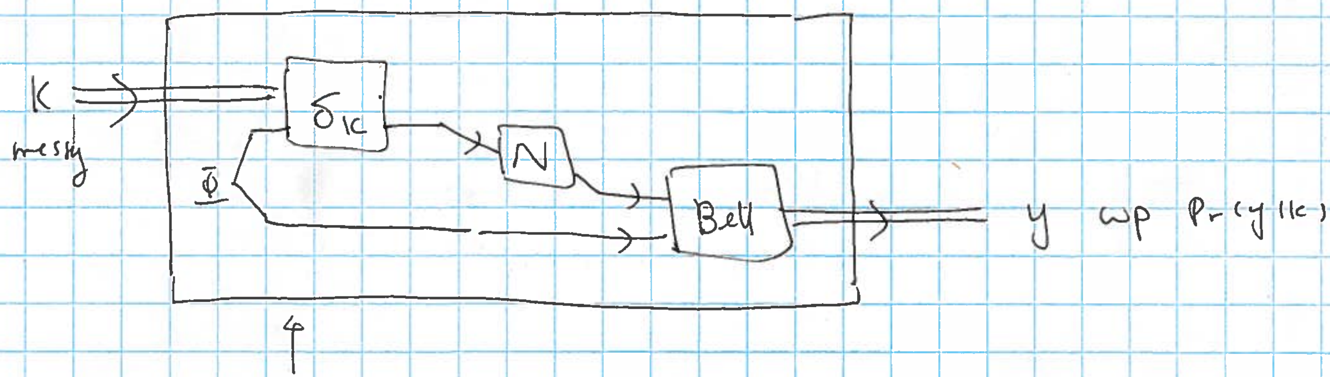
As $q \to 1$, $\frac{C_E(N)}{C(N)} \to d+1$ ∴ refuting both intuitive conjectures.

BSST 9904023 (C_E w/o BSST 2001 result).

"σ_k", "Bell" refer to d-dim gens.

Ideas:

(1) SD with noisy Q channel N. (or tele channel here)

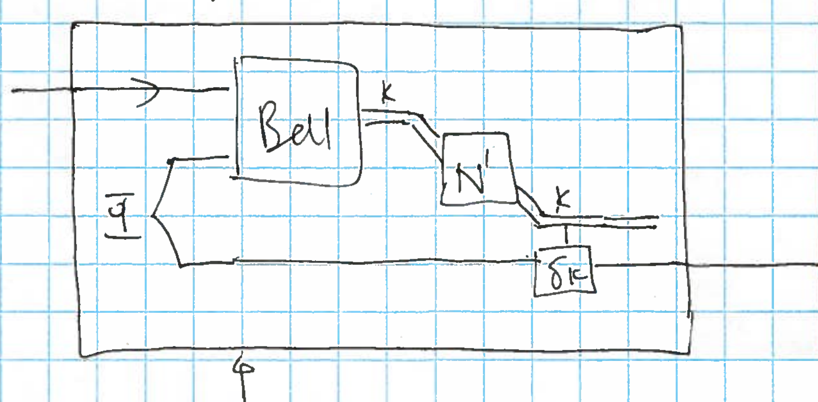


Net: classical channel N'

N' maps k $\begin{cases} \text{to } k & \text{wp } 1-f. \\ \text{to a random output (out of } d^2) & \text{wp } f. \end{cases}$

$$C_E(N) \geq C(N') = 2\log d - H_{d^2}\left(1 - f\left(\frac{d^2-1}{d^2}\right)\right)$$

(2) Simulating N with "teleportation based on N'":



Net: quantum channel N

$$C_E(N) \leq \#\text{ cbits used in implementing } N' \overset{\text{RST}}{\leq} C(N')$$

Remarks on C_E:

① $C_E(\mathcal{N}) = 0 \iff \mathcal{N}$ trivial $\iff \mathcal{N}(\rho)$ indep of ρ.

② We allow arbitrary $|\phi\rangle$ shared

but suffices to use ebits to achieve capacity.

③ $2Q_E = C_E$ $\quad (\forall \mathcal{N},\ 2Q_E(\mathcal{N}) = C_E(\mathcal{N}))$.

Pf ($\geq$) teleportation

($\leq$) super dense coding.

④ (Quantum) Reverse Shannon thm.

(•) Consider classical channels.

Think of direct coding as $\boxed{n N \underset{\text{approx simulation}}{\geq} n\, C(N) \cdot I}$ for large n.

The reverse shannon theorem says $\boxed{n\, C(N) \cdot I \geq n N \;+ \infty \text{ rbits}}$

BSST

Why such a perverted idea?

Then n_1 uses of N_1 simulates n_2 uses of N_2 for $\frac{n_1}{n_2} \approx \frac{C(N_2)}{C(N_1)}$

by $n_1 N_1 \geq n_1 C(N_1)\, I \geq \frac{n_1 C(N_1)}{C(N_2)} \cdot N_2$. $\therefore$ Channels are described by only 1 parameter $C(N)$!

(b) Quantum reverse Shannon theorem:

since 2000

reverse. (0912.5537 Bennett Devetak Harrow Shor Winter)

$n \, C_E(N)$ cbits. $\geq$ nN.

Alt pf 0912.3805 Berta, Christandl, Renner.

(*) + arbitrary entanglement

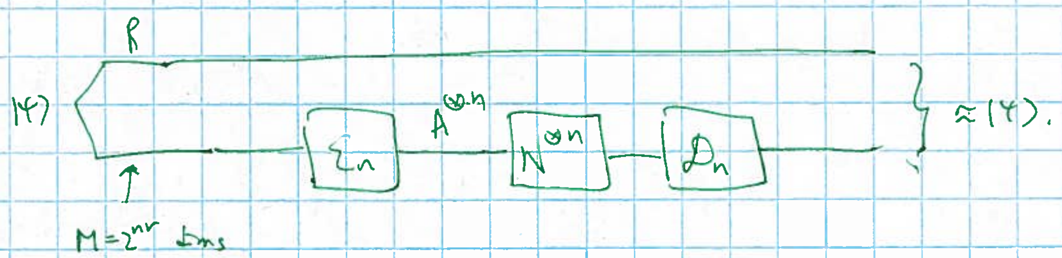
$n \, C_E(N)$ cbits $\leq$ nN + free ebits

BSST thm.

See guest lectures in S 2010.

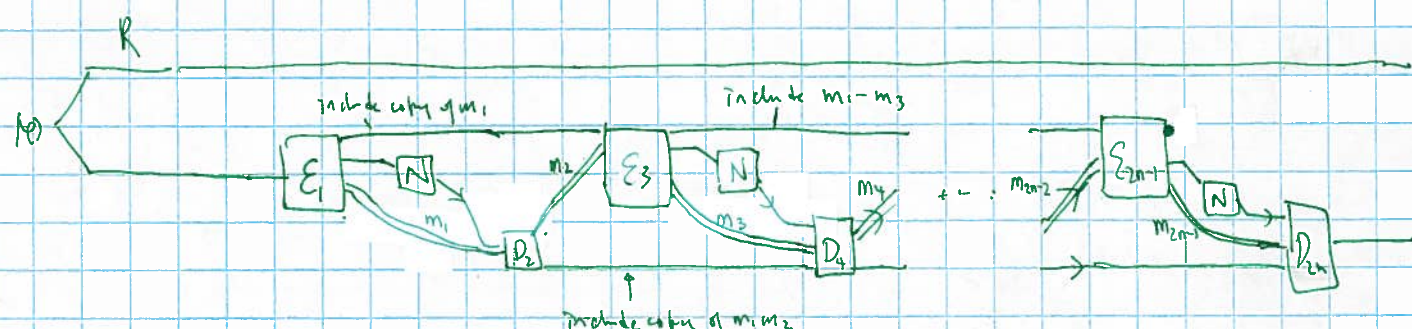
Now $Q_{\to}$, $Q_{\leftarrow}$, and $Q_{\leftrightarrow}$.   Motivation: N expensive
cc cheap.

- Unassisted:



- free forward cc:

- free 2-way cc:



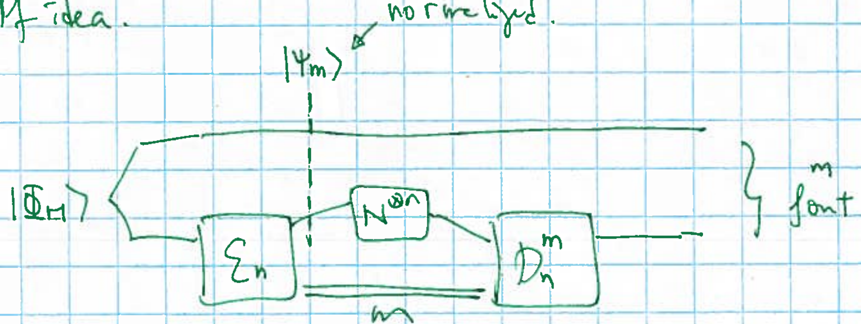
- free back cc

(no $m_1, m_3, \ldots, m_{2n-1}$ in above).

Thm 1 $\forall \mathcal{N}$, $Q(\mathcal{N}) = Q_{\rightarrow}(\mathcal{N})$.

ie free forward CC does not increase ASYMPTOTIC Q capacity.

Pf idea.



Idea: at least some outcomes are good & just "hard wire" one of them.

Output $\rho_{out} = \sum_m p_m \rho_{out}^m \approx |\Phi_m\rangle\langle\Phi_m|$

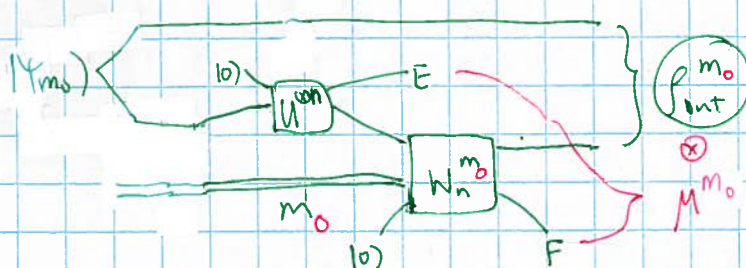
$\therefore \exists m_0$ s.t. $\rho_{out}^{m_0} \approx |\Phi_m\rangle\langle\Phi_m|$.

hardwire the good outcome.

If $\exists \mathcal{E}_n^{m_0}$ s.t.

then we're done. (Since m_0 need not be comm anymore.)

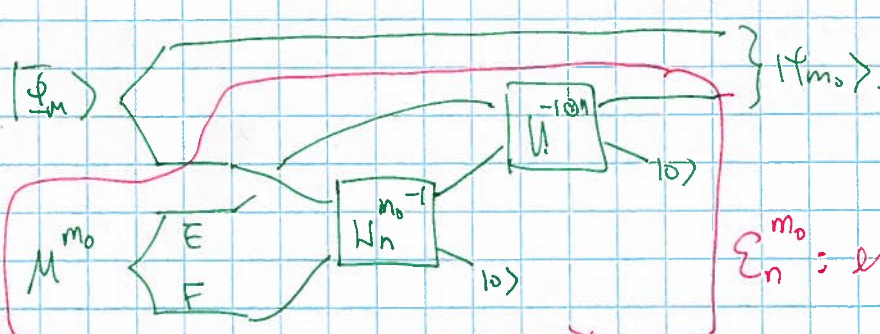
What is $|\psi_{m_0}\rangle$?



$\rho_{out}^{m_0} \approx |\Phi_m\rangle\langle\Phi_m|$.

$\otimes$ μ^{m_0}

NB. U, W_n^m iso exts of $\mathcal{N}$ & $\mathcal{D}_n^m$.



$\mathcal{E}_n^{m_0}$: encoding circuit

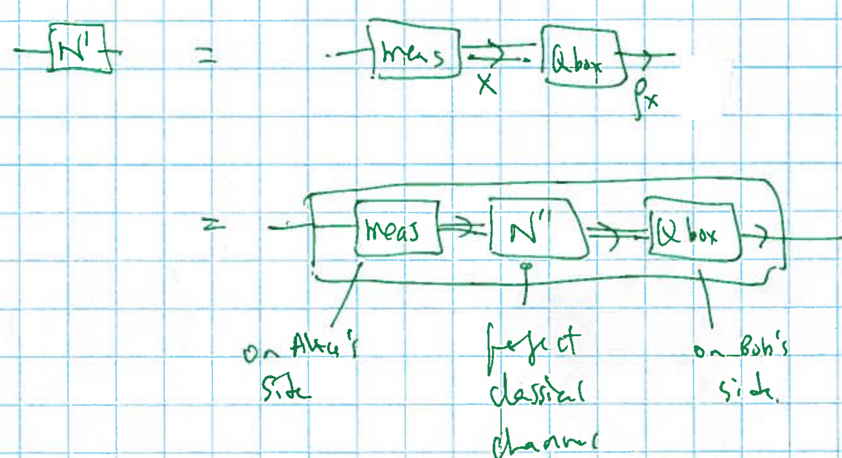
Cor (0311037 Kretschman & Werner)

If N' is an entanglement breaking channel

N is any channel

then $Q(N \otimes N') = Q(N)$

Pf. ① Recall characterization of ent breaking channel:



$$\therefore \quad N \otimes N' = (I \otimes \text{Qbox}) \circ (N \otimes N'') \circ (I \otimes \text{meas})$$

② Note also $Q(N_2 \circ N_1) \leq \min(Q(N_2), Q(N_1))$

(ex if haven't covered this before

ans in additive extension)

$$\therefore \quad Q(N \otimes N') \leq Q(N \otimes N'') \underset{\text{Thm.}}{=} Q(N)$$