Lec 19 Assisted capacities.

The HSW & LSD Thms give the unassisted classical & quantum capacities of a quantum channel $\mathcal{N}$. There, $\mathcal{N}$ is the only nonlocal resource available.

We can consider assisted capacities instead where some extra unlimited nonlocal resource is available for free.

e.g. $Q_{ss}(\mathcal{N})$ – q. capacity of $\mathcal{N}$ when "symmetric channels" (wrt B & E) are free

$C_E(\mathcal{N})$ – classical
$Q_E(\mathcal{N})$ – quantum
} capacity of $\mathcal{N}$ when entanglement is free.

$Q_{\rightarrow}(\mathcal{N})$
$Q_{\leftarrow}(\mathcal{N})$
$Q_{\leftrightarrow}(\mathcal{N})$
} quantum capacity of $\mathcal{N}$ when classical communication (forward / backward / 2-way) is free.

Assisting resources shouldn't trivialize the task:

eg Symmetric channels has $Q(A) = 0$

Entanglement is static

Classical communication has no quantum capacity.

Motivations -

- Operational interest (eg $Q \to (N)$)
- Simplified upper bounds
- Lends intuitions.
- Inspires unexpected results (superactivation)

Bennett Shor Smolin Thapliyal 0106052 (BSST):

$$C_E(N) = \max_{|\psi\rangle_{RA}} S(R{=}B)_{I\otimes N(|\psi\rangle\langle\psi|_{RA})}$$ ← 1-shot !

cf

$$C(N) = \max_{\rho_{in}=\sum_x p_x |x\rangle\langle x|} S(R{=}B)_{I\otimes N(\rho_{in})}$$ [Shannon]

① Additivity of the correlation expression.

Lemma:

Let $f(N) = \max_{|\psi\rangle_{RA}} S(R:B)_{I\otimes N(|\psi\rangle\langle\psi|)}$

Then $f(N_1 \otimes N_2) = f(N_1) + f(N_2)$

Pf. [$\geq$] If $\rho_{R_1R_2B_1B_2} = \rho_{R_1B_1} \otimes \rho_{R_2B_2}$

Then $S(R_1R_2:B_1B_2)$

$= S(R_1R_2) + S(B_1B_2) - S(R_1R_2B_1B_2)$

By property of state $= S(R_1) + S(R_2) + S(B_1) + S(B_2) - [S(R_1B_1) + S(R_2B_2)]$

$= S(R_1:B_1) + S(R_2:B_2)$

$\therefore$ If $|\psi_1\rangle_{R_1A_1}$ & $|\psi_2\rangle_{R_2A_2}$ are optimal for $f(N_1), f(N_2)$

Then $f(N_1 \otimes N_2)$

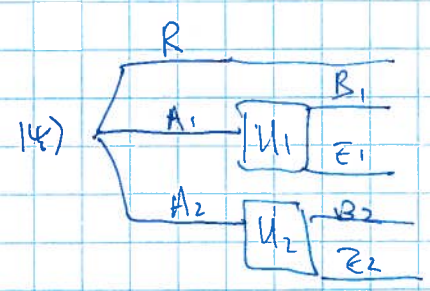
$\geq S(R_1R_2:B_1B_2)_{I\otimes N_1\otimes N_2(|\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|)}$

($|\psi_1\rangle\langle\psi_1|$: R_1A_1, $|\psi_2\rangle\langle\psi_2|$: R_2A_2; I: R_1R_2)

$= I\otimes N_1(|\psi_1\rangle\langle\psi_1|) \otimes I\otimes N_2(|\psi_2\rangle\langle\psi_2|)$

$= S(R_1:B_1)_{I\otimes N_1(|\psi_1\rangle\langle\psi_1|)} + S(R_2:B_2)_{I\otimes N_2(|\psi_2\rangle\langle\psi_2|)}$

[$\leq$]: Let U_1, U_2 be TSO ext of N_1, N_2



$$f(N_1 \otimes N_2) = \max_{|\psi\rangle_{RA_1A_2}} S(R:B_1B_2)_{I \otimes N_1 \otimes N_2 (|\psi\rangle\langle\psi|)}$$

Evaluate $S(R:B_1B_2)$ on $I \otimes U_1 \otimes U_2 (|\psi\rangle\langle\psi|)$:

$$= S(R) + S(B_1B_2) - S(RB_1B_2)$$

$$= S(B_1B_2E_1E_2) + S(B_1B_2) - S(E_1E_2)$$

$$= S(B_1B_2|E_1E_2) + S(B_1B_2)$$

$$\leq S(B_1|E_1) + S(B_2|E_2) + S(B_1B_2) \quad (\star)$$

lemma proved in additivity of coh info for deg channels

$$\overset{SA}{\leq} S(B_1|E_1) + S(B_2|E_2) + S(B_1) + S(B_2)$$

$$= \left[S(B_1E_1) - S(E_1) + S(B_1)\right] + \left[S(B_2E_2) - S(E_2) + S(B_2)\right]$$

$$= \left[S(\underbrace{RB_2E_2}_{R_1}) + S(B_1) - S(R\underline{B_1}B_2E_2)\right] + \underbrace{[\quad]}_{\text{Similarly, } R_2 = RB_1E_1}$$

$$= S(R_1:B_1) + S(R_2:B_2)$$

$S(R_1:B_1)$ evaluated on $I \otimes U_1 \otimes U_2 (|\psi\rangle\langle\psi|)$; $S(R_2:B_2)$ evaluated on $I \otimes U_1 \otimes U_2 (|\psi\rangle\langle\psi|)$

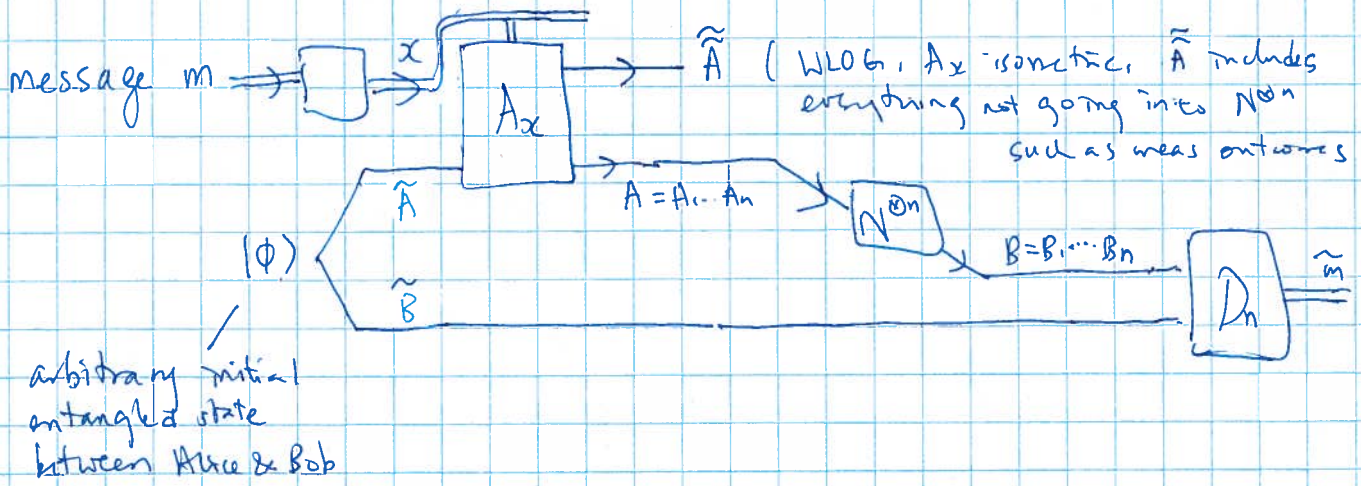
$\leq f(N_1) + f(N_2)$.

$\therefore f(N_1 \oplus N_2) = \max_{\psi \in \mathrm{PA}_1 A_2} [\ \ldots\ ]$

$\leq f(N_1) + f(N_2)$

(2) Defining the capacity in terms of (M, n) code as before.

The most general code has the form:



Unassisted: without $\tilde{A}$, $\tilde{B}$, $\tilde{\tilde{A}}$.

Lack of feedback from Bob to Alice simplifies the structure of the code.

Same: rate: $\frac{1}{n} \log M$, error if $\tilde{m} \neq m$, achievable rate as before.

(3) Converse (of PSST):

(a) Sufficiency of isometric encoding ie WLOG, no $\hat{\hat{A}}$

(b) Converse without $\tilde{\tilde{A}}$.

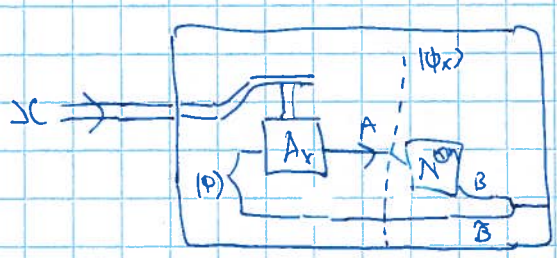
Pf (a) Let R denote the completely randomizing map.

Since R can be produced locally:

$\mathcal{E}(\mathcal{N} \otimes R) = \mathcal{E}(\mathcal{N})$.

If we consider $\mathcal{N} \otimes R$, the encoding is isometric.

(b). The most general protocol involves a Q-box:



$\rho_{x\tilde{B}} = I \otimes \mathcal{N}^{\otimes n} \cdot A_x(|\phi\rangle\langle\phi|)$

By the capacity thm of the Q-box

Classical capacity $= \sup_n \frac{1}{n} \max_{A_x, p_x} \chi(p_x, \rho_x)$

Useful facts (to be used later):

$\forall x, \quad \rho_{x\tilde{B}} = \mathrm{tr}_{\tilde{A}} |\phi\rangle\langle\phi|$ induced by x.

$S(\rho_{x\tilde{B}}) = S(\mathrm{tr}_{\tilde{A}} |\phi\rangle\langle\phi|) = S(\mathrm{tr}_{\tilde{B}} |\phi_x\rangle\langle\phi_x|)$.

↳ requires A_x unitary

Lemma: Let ρ_0, ρ_1 be density matrices.

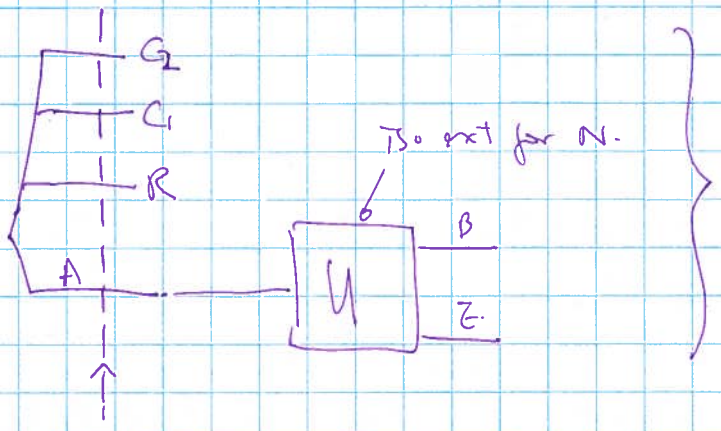
$0 \le p_0 \le 1$, $p_1 = 1 - p_0$, $\rho = p_0\rho_0 + p_1\rho_1$

Let $\psi_{\rho_0}, \psi_{\rho_1}, \psi_\rho$ purify ρ_0, ρ_1, and ρ.

Then:

$$S(\rho) - S(I \otimes \mathcal{N}(\psi_\rho)) \geq \sum_{i=0}^{1} p_i \left[S(\rho_i) - S(I \otimes \mathcal{N}(\psi_{\rho_i})) \right] \quad \star$$

Pf: Consider the state:



$$|\psi\rangle = \left[\sqrt{p_0}\,|00\rangle\,|\psi_{\rho_0}\rangle + \sqrt{p_1}\,|11\rangle\,|\psi_{\rho_1}\rangle \right]_{C_2 C_1 R A}$$

↑ (is there to make C_1 classical)

- $S(\rho) = S(A) = S(C_2 C_1 R)_{|\alpha\rangle\langle\alpha|} = S(C_2 C_1 R)_{|\beta\rangle\langle\beta|}$
- $S(I \otimes \mathcal{N}(\psi_\rho)) = S(C_2 C_1 R B)_{|\beta\rangle\langle\beta|}$

} + LHS of $\star$.

- $S(C_1 R)_{|\alpha\rangle\langle\alpha|} = S(C_1 R)_{|\beta\rangle\langle\beta|} = H(\{p_0, p_1\}) + \sum_{i=0}^{1} p_i S(\rho_i)$
- $S(C_1 R B)_{|\beta\rangle\langle\beta|} = H(\{p_0, p_1\}) + \sum_{i=0}^{1} p_i S(I \otimes \mathcal{N}(\psi_{\rho_i}))$

} − RHS of $\star$.

$\therefore$ LHS - RHS of $\star$

$= \left[S(C_2 C_1 R) - S(C_2 C_1 R B) \right] - \left[S(C_1 R) - S(C_1 R B) \right]$ all ex-th-1 on $|RX|$

$= S(\underline{C_1 R}\ \underline{C_2}) + S(\underline{C_1 R}\ \underline{B}) - S(\underline{C_1 R}) - S(\underline{C_1 R}\ \underline{B}\ \underline{C_2})$

≥ 0 by SSA. on $C_1 R$, B, C_2.

Pf of (b):

$$\chi\left(\{p_x,\ \rho_{xB\tilde{B}}\}\right)$$

$$= S\left(\sum_x p_x \rho_{xB\tilde{B}}\right) - \sum_x p_x S\left(\rho_{xB\tilde{B}}\right)$$

$$\overset{SA}{\leq} S\left(\sum_x p_x \rho_{xB}\right) + S\left(\sum_x p_x \rho_{x\tilde{B}}\right) - \sum_x p_x S\left(\rho_{xB\tilde{B}}\right)$$

(x indexes the first sum)

$$= \sum_x p_x S\left(\rho_{x\tilde{B}}\right)$$

$$= \sum_x p_x S\left(\mathrm{tr}_{\tilde{B}}\left(\psi_x X \psi_x\right)\right)$$

(the input into $N^{\otimes n}$; output of $I\otimes N^{\otimes n}$)

$$\overset{\text{lemma}}{\leq} S\left(\sum_x p_x \rho_{x\tilde{B}}\right) - S\left(\sum_x p_x \rho_{x\tilde{B}B}\right)$$

$$= \left[S(B) + S(\tilde{B}) - S(B\tilde{B})\right] = S(\tilde{B}:B)$$

($\tilde{B}$: R; B: $B_1 B_2 \cdots B_n$)

Evaluated on $\sum_x p_x \rho_{x\tilde{B}B} = \sum_x p_x\ I\otimes N^{\otimes n} \circ A_x\left(|\phi\rangle\langle\phi|\right)$

$$= I\otimes N^{\otimes n}\left(\sum_x p_x\ I\otimes A_x\ |\phi\rangle\langle\phi|\ I\otimes A_x^\dagger\right)$$

$$= I\otimes N^{\otimes n}(\rho).$$

Sub over n, A_x, $|\psi\rangle$

$$\therefore (E|N) \leq \uparrow \frac{1}{n} \chi(\{p_x, \rho_x \otimes \hat{R}\})$$

$$\leq \frac{1}{n} f(N^{\otimes n})$$

$$= f(N)$$

by additivity

④ Direct coding. (Sketch).

Note we want to show

$$C_E(N) \geq \max_{|\psi\rangle_{RA}} S(R\rangle B) \quad I\otimes N(|\psi\rangle\langle\psi|)$$

$$= \max_{\mu} S(\mu) + S(N(\mu)) - S(I\otimes N(\psi_\mu))$$

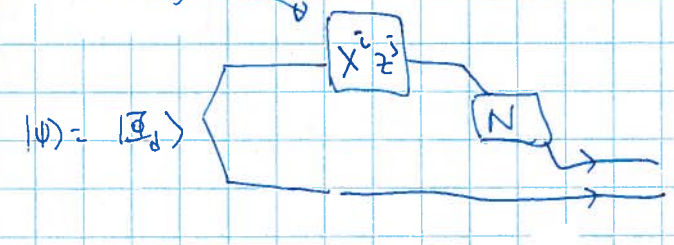
μ ↑ input state

$\mu \leftrightarrow \mathrm{tr}_R |\psi\rangle\langle\psi|_{RA}$; $S(R) = S(A)$; $S(B)$; $S(RB)$; ψ_μ ↑ purification of μ

Note: all expressions independent of which purification.

- Case 1: If $\mu = \frac{I}{d_A}$, $d = d_A$, $X|u\rangle = |u+1 \bmod d\rangle$, $Z|u\rangle = \omega^u|u\rangle$ where $\omega^d = 1$.

d^2 possible x's: (i,j)



$p_x = \frac{1}{d^2}$.

Then $\chi(\{p_x, \rho_x\}) = S\left(\sum_x p_x \rho_x\right) - \sum_x p_x S(\rho_x)$

$S(\rho_x) = S((X^i Z^j)^\dagger \otimes N(\Phi_d)) = S(I\otimes N(\Phi_d))$.

$\sum_x p_x \rho_x = \sum_x p_x\, I\otimes N(I\otimes X^i Z^j \Phi_d I\otimes Z^j X^i)$

$= I\otimes N\left(\sum_{ij} \frac{1}{d^2} I\otimes X^i Z^j \Phi_d I\otimes Z^j X^i\right)$

$= I\otimes N\left(\frac{I}{d}\otimes\frac{I}{d}\right) = \frac{I}{d}\otimes N\left(\frac{I}{d}\right)$.

$= S\left(\frac{I}{d}\right) + S\left(N\left(\frac{I}{d}\right)\right) - S(I\otimes N(\Phi_d))$.

- Case 2 : μ = projector onto a subspace

  [Essentially same as before].

- Case 3 : arbitrary μ.

Idea: Consider n-shot expression.

$$S(\underbrace{\mu^{\otimes n}}) + S(\underbrace{N^{\otimes n}(\mu^{\otimes n})}) - S\Big(\underbrace{I\otimes N^{\otimes n}(\psi_\mu^{\otimes n})}\Big)$$

$\approx$ projector Π onto the strongly typical subspace.

$\approx \tau$

use instead the purification of τ

Reduces to case 2. with $S(\mu^{\otimes n}) \approx n S(\mu)$

$S(N^{\otimes n}(\mu^{\otimes n})) \approx n\, S(N(\mu))$

$S(I\otimes N^{\otimes n}(\psi_\mu)) \approx n S(I\otimes N(\psi_\mu))$.

eg. 1 If $N = I$, $|\psi\rangle_{RA} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ optimal

$C_E(N) = 2$ superdense coding

eg 2. If $N = N_p$ erasure channel w/ error p.

$$I \otimes N_1(|\psi\rangle\langle\psi|_{RA}) = (1-p)|\psi\rangle\langle\psi|_{RB} + p\left(\mathrm{tr}_B |\psi\rangle\langle\psi|\right)_R \otimes |2\rangle\langle 2|_B$$

$S(R=B)$

$I \otimes N_1(|\psi\rangle\langle\psi|_{RA})$

$$= S(R) + S(B) - S(RB)$$

$$= S(\mathrm{tr}_B |\psi\rangle\langle\psi|) + [H(p) + (1-p) S(\mathrm{tr}_R |\psi\rangle\langle\psi|) + p \cdot 0]$$
$$- [H(p) + (1-p)\cdot 0 + p\, S(\mathrm{tr}_B(|\psi\rangle\langle\psi|))]$$

$$= (1+1-p-p)\, S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$$

$$= 2(1-p)\, S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$$

$$\therefore C_E(N_p) = \max_{|\psi\rangle_{RA}} 2(1-p)\, S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$$

$$= 2(1-p) \quad \left(\text{when } |\psi\rangle_{RA} = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right).$$

eg 3. N classical, then $C_E(N) = C(N)$. (Ex.)

eg 4. N_q = depolarizing channel. As $q \to 1$, $C(N), C_E(N) \to 0$ but $C_E(N) = C(N) \to \infty$.

$N_q(\rho) = (1-q)\rho + q\frac{I}{2}$.

Remarks on C_E:

① $C_E(N) = 0 \iff N$ trivial $\iff N(\rho)$ indep of ρ.

② We allow arbitrary $|\phi\rangle$ shared but suffices to use qbits to achieve capacity.

③ $2QE = C_E$ $\quad (\forall N,\ 2QE(N) = C_E(N))$.

Pf $\geq$ teleportation

$\leq$ Swift dense coding.

④ (Quantum) Reverse Shannon thm.

⑤ Consider classical channels.

Think of direct coding as $nN \geq n C(N) \cdot I$ for large n.

↓ approx simulation

The reverse Shannon theorem says $n C(N) \cdot I \geq nN$ + ∞ cbits

↓

BSST.

Why such a perverted idea?

Then n_1 uses of N_1 simulates n_2 uses of N_2 for $\frac{n_1}{n_2} \approx \frac{C(N_2)}{C(N_1)}$

by $n_1 N_1 \geq n_1 C(N_1) I \geq \frac{n_1 C(N_1)}{C(N_2)} \cdot N_2$. ∴ Channels are described by only 1 parameter $C(N)$!

(b) Quantum reverse Shannon theorem:

since 2000

reverse. (0912.5537 Bennett Devetak Harrow Shor Winter)

n ($\frac{1}{2}I(A)$ cbits. $\geq$ nN.

(*) + arbitrary entanglement

At pt 0-912.3805 Berta, Christandl, Renner.

n ($\frac{1}{2}I(A)$ cbits $\leq$ nN + free ebits

$\Downarrow$

BSST thm.

See guest lectures in S 2010.