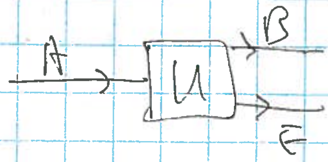
CO781 / QIC 890, Lec 18, Nov 15, 2016.

Consequences of the LSD thm (part I / III)

- Complementary channel
- Degradable, anti degradable channels.
- The erasure channel (as main example throughout)

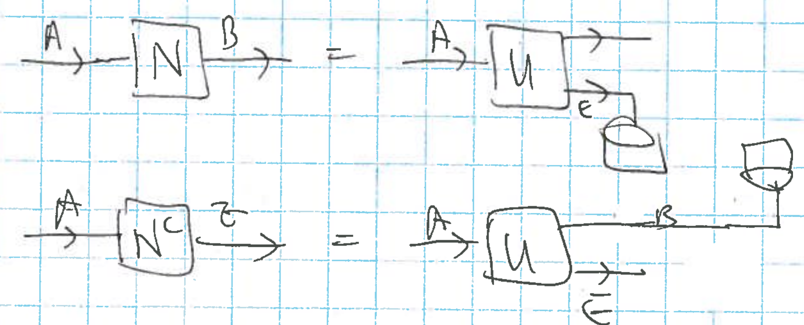
Complementary Channel:

Let N be a channel, and U its isometric extension.



The complementary channel N^c is given by:

$$N^c(\rho) = \mathrm{tr}_B(U \rho U^\dagger)$$



Both U & N^c determined up to a unitary on E.

$(N^c)^c = N$ up to a unitary on B.

eg. Binary erasure channel

$$\mathcal{N}_p(\rho) = (1-p)\,\rho + p\,|2\rangle\langle 2|$$

where $\rho \in B(\mathbb{C}^2)$ and $|2\rangle$ ortho to the input space thereby signifying an error

Denote the output space B_1.

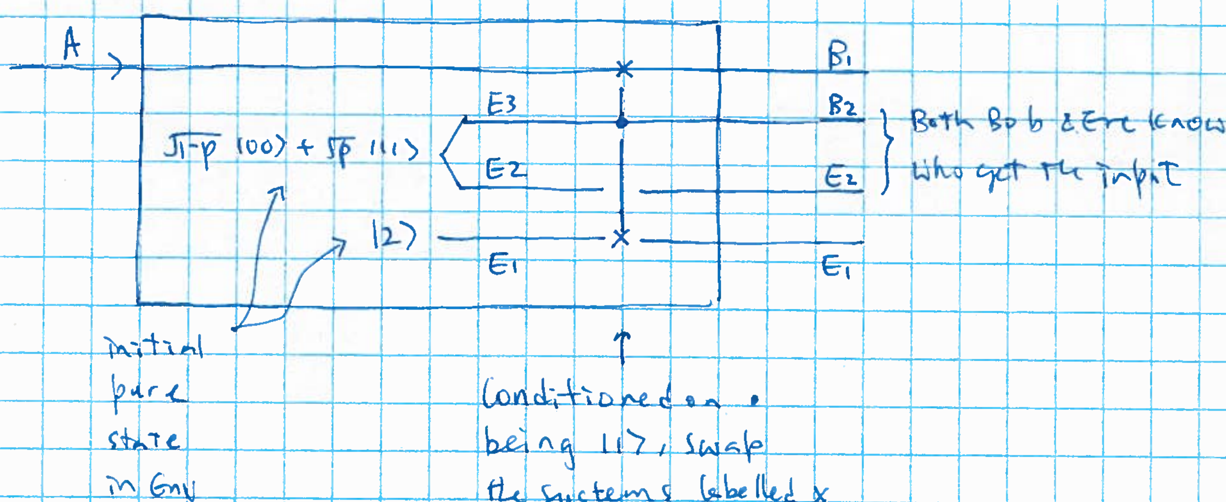
Let Bob attach $|0\rangle$ on B_2, and apply the unitary

$$(|0\rangle\langle 0| + |1\rangle\langle 1|)_{B_1} \otimes I_{B_2} + |2\rangle\langle 2|_{B_1} \otimes \sigma_{x\,B_2}$$

The channel output is reversibly mapped to:

reuse the symbol. → $$\mathcal{N}_p(\rho) = (1-p)\,\rho_{B_1} \otimes |0\rangle\langle 0|_{B_2} + p\,|2\rangle\langle 2|_{B_1} \otimes |1\rangle\langle 1|_{B_2}$$

Thus the following is an isometric extension of $\mathcal{N}_p$:



$$* \quad \mathcal{N}_p^c = \mathcal{N}_{1-p} \; !$$

A useful digression

0605009 Kretschmann, Schlingemann, (R) Werner

① Continuity of Stinespring's dilations (Thm 1).

For any 2 channels N_1, N_2:

$$\inf_{U_1, U_2} \| U_1 - U_2 \|_\infty^2 \le \| N_1 - N_2 \|_\diamond \le 2 \inf_{U_1, U_2} \| U_1 - U_2 \|_\infty$$

Where the inf is taken over the dilations U_1, U_2 of N_1, N_2.

? or min

identity completely randomizing channel.

② Approx complementary relation between I & R (Thm 3)

$$\forall N, \quad \frac{1}{4} \inf_D \| D \circ N - I \|_\diamond^2 \le \| N^c - R \|_\diamond \le 2 \inf_D \| D \circ N - I \|_\diamond^{\frac{1}{2}}$$

where the inf is taken over decoding maps from dout to din dims.

Degradable & anti degradable channels:

sometimes called the degrading map.

Def: N is called degrable if $\exists D$ (TCP map)

s.t. $D \circ N = N^c$

Def: N is called anti degrable if N^c is degradable.

Intuition: if N is degradable, "Bob is better than Eve".

so . . . anti . . . . worse . . . .

NB: The decoding map D exchanges N & N^c.

Def: N is called "symmetric" if N is both degrable & anti degradable.

eg. Let N_p = erasure channel with error prob p.

Then N_p is degradable for $p \leq \frac{1}{2}$

- - - - anti deg - - $p \geq \frac{1}{2}$.

Obs $N_p^c = N_{1-p}$ (use iso ext shown earlier)

Obs : $N_q \circ N_p = N_{p+q-pq}$ (Here N_q has input space spanned by $|0\rangle, |1\rangle, |2\rangle$.)

Pf : $N_q \circ N_p(\rho)$

$= N_q((1-p)\rho + p|2\rangle\langle 2|)$

$= (1-q)[(1-p)\rho + p|2\rangle\langle 2|] + q|2\rangle\langle 2|$.

$= (1-q)(1-p)\rho + (p+q-pq)|2\rangle\langle 2|$.

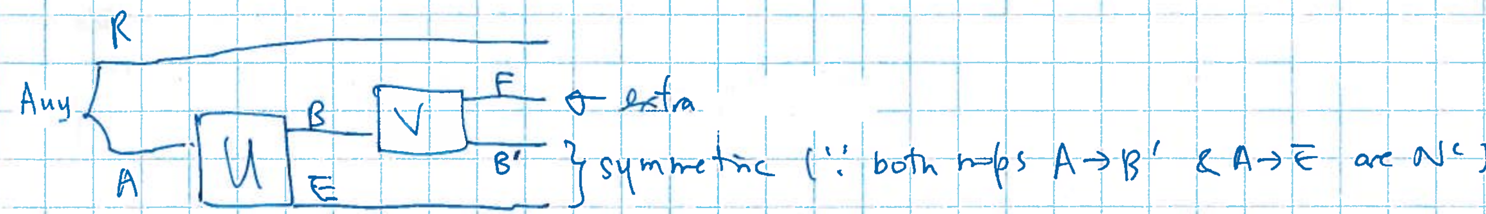
<u>Cor</u> : If $p \leq \frac{1}{2}$, let $q = \frac{1-2p}{1-p}$ ← nonneg if $p \leq \frac{1}{2}$

then $p+q-pq = p + q(1-p)$

$= p + 1 - 2p = 1-p$.

then $N_q \circ N_p = N_{1-p}$ ∴ N_p degradable.

Another characterization of degradable channels:



where V = isometric extension of the degrading map

NB: V transmits F from Bob to Eve.

NB: F belongs to Eve for anti deg channels.

NB: F trivial if N symmetric

Thm If N antidegradable, then $Q(N)=0$.

Thm' In fact, not a single qubit can be sent with arbitrarily large number of uses of N.

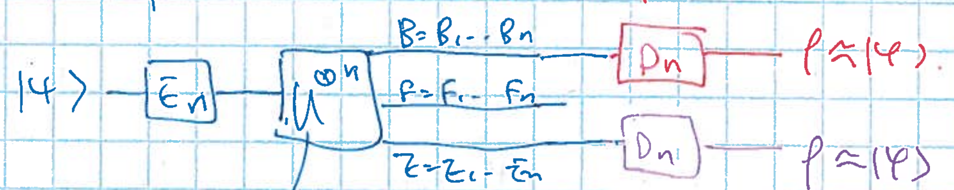
"Pf:" If Alice can send q data to Bob, Eve can get a copy too, implying cloning.

Pf: If $|\psi\rangle \in \mathbb{C}^d$ can be transmitted with high fidelity, with n uses of N,

$\exists\, E_n, D_n$ s.t. $|\psi\rangle \to \boxed{E_n} - \boxed{N^{\otimes n}} - \boxed{D_n} \to \rho \approx |\psi\rangle$.

This implies:

hypothesis:



consequence

isometric extension of N that incorporates the degrading isometry for Eve.

Thus cloning with high fidelity – a contradiction.

Example: Classical channel is symmetric (cf A1)

Example: $Q(N_p)=0$ $\forall p \geq \frac{1}{2}$ in erasure channels.

Theorem [Devetak & Shor 0311131]

If N_1, N_2 are degradable, then $Q^{(1)}(N_1 \otimes N_2) = Q^{(1)}(N_1) + Q^{(1)}(N_2)$.

Corollary:

If N is degradable, then $Q(N) = Q^{(1)}(N)$. ($\because Q^{(r)}(N) = Q^{(1)}(N) \ \forall r$)

Lemma 1: Let $R = R_1R_2$, $B = B_1B_2$, $\rho_{RB} = \sigma_{R_1B_1} \otimes \varsigma_{R_2B_2}$

then $I_c(R \rangle B)_\rho = I_c(R_1 \rangle B_1)_\sigma + I_c(R_2 \rangle B_2)_\varsigma$

Pf (Straightforward, omitting the state labels):

$$I_c(R \rangle B) = S(B) - S(RB)$$

$$= S(B_1B_2) - S(R_1R_2B_1B_2)$$

product state

$$= S(B_1) + S(B_2) - S(R_1B_1) - S(R_2B_2)$$

$$= I_c(R_1 \rangle B_1) + I_c(R_2 \rangle B_2)$$

Lemma 2: For any states in 4 systems $STXY$

$$S(ST|XY) \leq S(S|X) + S(T|Y)$$

Pf: RHS − LHS

$$= S(S|X) + S(T|Y) - S(ST|XY)$$

$$= S(SX) - S(X) + S(TY) - S(Y) - S(STXY) + S(XY)$$

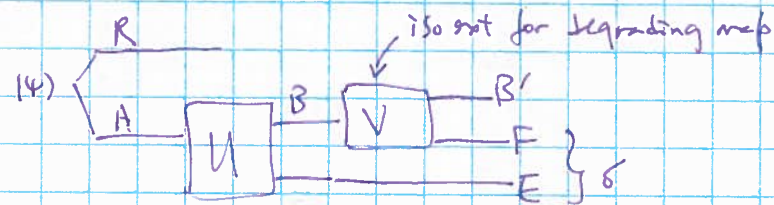
$$= S(SX : TY) - S(X:Y) \geq 0$$ by monotonicity of QMI

(*) <u>Lemma 3</u>: If N degradable with iso ext U, $|\psi\rangle_{RA}$ any input

then $I_c(R\rangle B)_{I \otimes N(|\psi\rangle\langle\psi|)}$

$$= S(F|E)_{\mathrm{Tr}_{RB'}(V \otimes I_{RE})(U \otimes I_R)|\psi\rangle\langle\psi|(U^\dagger \otimes I_R)(V^\dagger \otimes I_{RE})} =: \sigma$$

where



with B' & E symmetric.

<u>Pf</u>: $I_c(R\rangle B)_{I \otimes N(|\psi\rangle\langle\psi|)}$

$= S(B) - S(E)$

V iso $= S(B'F) - S(E)$

B', E sym $= S(EF) - S(E)$

$= S(F|E)_\sigma$

NB $I_c(R\rangle B) = -S(R|B)$

The above gives "opp" properties in this special case (N deg).

Pf of theorem:

[$\geq$] Let $|\psi_i\rangle$ attain the max of $I_c(R_i \rangle B_i)$ for N_i, $i=1,2$.

Then $Q^{(1)}(N_1 \otimes N_2)$

$$\geq I_c(R_1R_2 \rangle B_1B_2)_{\underbrace{I_{R_1R_2} \otimes N_1 \otimes N_2 (|\psi_1\rangle\langle\psi_1|_{R_1A_1} \otimes |\psi_2\rangle\langle\psi_2|_{R_2A_2})}_{\text{product over } R_1B_1 \ \& \ R_2B_2}}$$

lemma 1

$$= I_c(R_1 \rangle B_1)_{I_{R_1} \otimes N_1 (|\psi_1\rangle\langle\psi_1|)} + I_c(R_2 \rangle B_2)_{I_{R_2} \otimes N_2 (|\psi_2\rangle\langle\psi_2|)}$$

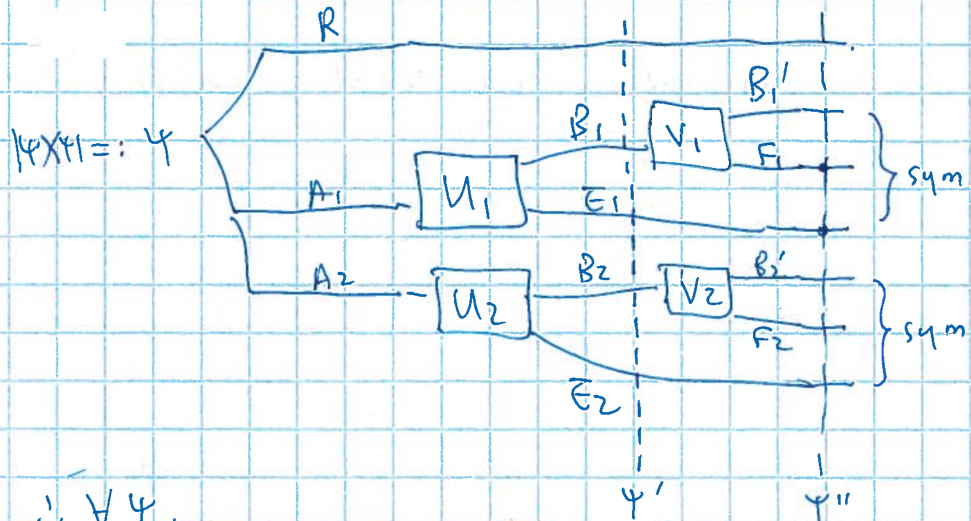
$$= Q^{(1)}(N_1) + Q^{(2)}(N_2).$$

optimality of $|\psi_1\rangle, |\psi_2\rangle$

Pf [$\leq$]:

Suppose U_1, V_1 are iso exts of N_1 & its degrading map
$U_2\ V_2$ - - N_2 - -

Then $U_1 \otimes U_2$, $V_1 \otimes V_2$ - - - - $N_1 \otimes N_2$ - - - - - -



$(B_1' B_2'$ sym $E_1 E_2)$

$\therefore \forall \psi$,

$I_c(R > B_1 B_2)_{\psi'}$

Lemma 3

$= S(F_1 F_2 | E_1 E_2)_{\psi''}$

Lemma 2

$\leq S(F_1 | E_1)_{\psi''} + S(F_2 | E_2)_{\psi''}$

$= I_c(\underbrace{R B_2 E_2}_{R_1} > B_1)_{\psi'} + I_c(\underbrace{R B_1 E_1}_{R_2} > B_2)_{\psi'}$

$\leq Q^{(1)}(N_1) + Q^{(1)}(N_2)$

$\therefore Q^{(1)}(N_1 \otimes N_2) \leq Q^{(1)}(N_1) + Q^{(1)}(N_2)$

Example: $Q(N_p) = Q^{(1)}(N_p)$ if $p \le \frac{1}{2}$ (so N_p degradable)

N_p: erasure channel

Useful result on $I_c(R\rangle B)$ if part of B is classical:

Let $B = B_1 B_2$, $\rho_{RB} = \sum_i p_i \, \rho_{RB_1}^{(i)} \otimes |i\rangle\langle i|_{B_2}$

Then $I_c(R\rangle B)_\rho = \sum_i p_i \, I_c(R\rangle B_1)_{\rho_{RB_1}^{(i)}}$.

Pf: $\rho_B = \sum_i p_i \, \rho_{B_1}^{(i)} \otimes |i\rangle\langle i|_{B_2}$ where $\rho_{B_1}^{(i)} = \mathrm{tr}_R \, \rho_{RB_1}^{(i)}$

$S(RB) = H(p) + \sum_i p_i \, S(RB_1)_{\rho_{RB_1}^{(i)}}$

$S(B) = H(p) + \sum_i p_i \, S(B_1)_{\rho_{B_1}^{(i)}}$

$$
\begin{aligned}
I_c(R\rangle B)_\rho &= S(B) - S(RB) \\
&= \sum_i p_i \underbrace{S(B_1)_{\rho_{B_1}^{(i)}}} - \sum_i p_i \underbrace{S(RB_1)_{\rho_{RB_1}^{(i)}}} \\
&= \sum_i p_i \, I_c(R\rangle B_1)_{\rho_{RB_1}^{(i)}}
\end{aligned}
$$

Back to N_p: $\forall \, |\psi\rangle_{RA}$ with $\psi_R = \mathrm{tr}_A |\psi\rangle\langle\psi|$

$$
I_R \otimes N(|\psi\rangle\langle\psi|) = (1-p)\, |\psi\rangle\langle\psi|_{RB_1} \otimes |0\rangle\langle 0|_{B_2} + p \, \psi_R \otimes |2\rangle\langle 2|_{B_1} \otimes |1\rangle\langle 1|_{B_2}
$$

$$
\begin{aligned}
\therefore I_c(R\rangle B) &= (1-p)\, I_c(R\rangle B_1)_{|\psi\rangle_{RB_1}} + p \, I_c(R\rangle B_1)_{\psi_R \otimes |2\rangle\langle 2|_{B_1}} \\
&= (1-p) \underset{(S_B = S(\psi_R),\ S_{RB} = 0)}{S(\psi_R)} + p\,[0 - S(\psi_R)] = (1-2p)\, S(\psi_R).
\end{aligned}
$$

$\therefore$ If $p \le \frac{1}{2}$, $Q^{(1)}(N) = \max\limits_{|\psi\rangle} (1-2p)\, S(\psi_R) = 1-2p$.

If $p > \frac{1}{2}$, N_p antidegradable & $Q(N) = 0$.

Together $Q(N_p) = \max(1-2p, 0)$.

Note: note $Q(N_p)$ & $Q^{(1)}(N_p)$ cts in p

but optimal $|\psi\rangle_{RA}$ jumps from the max entangled state

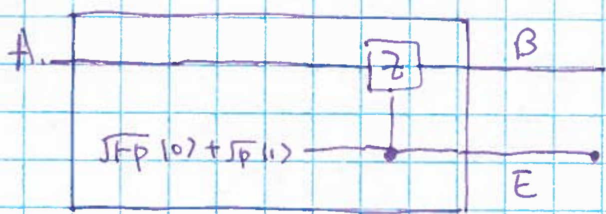
to a product state as p increases pass $\frac{1}{2}$.

Example 2: Phase damping channel

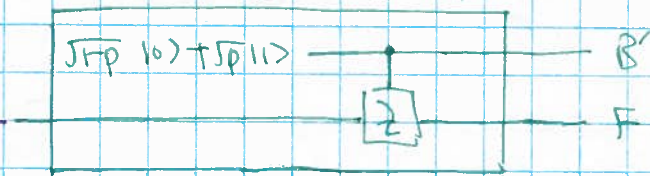
$$N_p(\rho) = (1-p)\,\rho + p\, Z \rho Z^{\dagger}, \qquad \rho \in B(\mathbb{C}^2), \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- To see N_p is degradable:

Note this isometric extension:



and this degrading map:



Degrading map is valid since it commutes with N_p so B' & E are symmetric

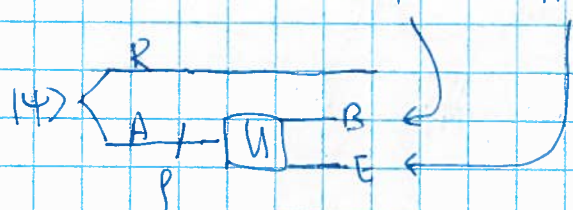
$\therefore$ N_p degradable $\forall p \in [0,1]$, worse case $p = \frac{1}{2}$.

- To find $Q^{(1)}(N_p)$:

$*$ Let $Q^{(1)}(N) = I_c(R\rangle B)_{I\otimes N(|\psi\rangle\langle\psi|)}$ for some $|\psi\rangle$ attaining the max.

$= [S(B) - S(E)]_{(I\otimes U)|\psi\rangle}$ for U = isometric extension of N

$= S(B)_{N(\rho)} - S(E)_{N^c(\rho)}$ where $\rho = \mathrm{tr}_R |\psi\rangle\langle\psi|$



* We now prove that WLOG, ρ is diagonal in the computation basis (defined by Z in N_ρ).

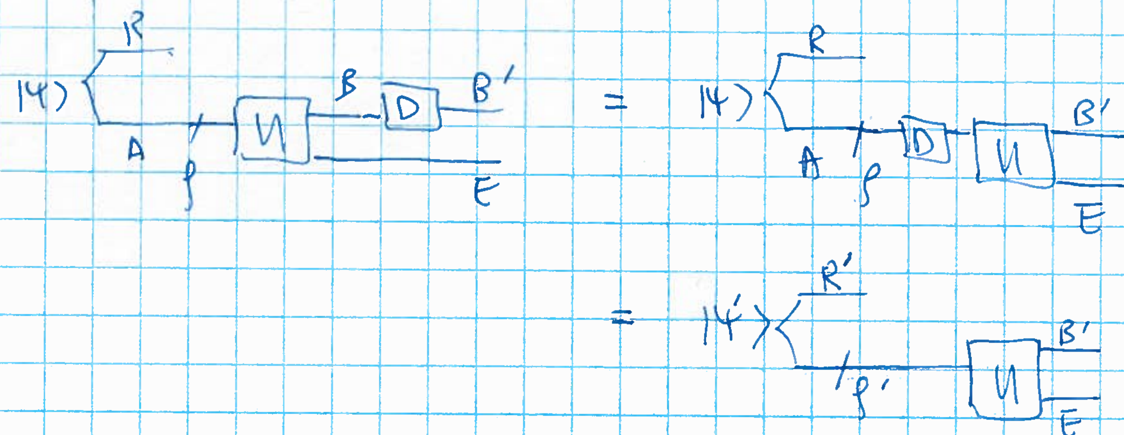
* Let $D(\rho) = \text{Diag}(\rho) = \frac{1}{2}[I\rho I + Z\rho Z]$

  Note D is unital (ie $D(I) = I$)).

  So $\forall \rho$, $S(D(\rho)) \geq S(\rho)$. $(\neq)$

* If we apply D to B, it commutes with the action of U.

  So:



where $\rho' = D(\rho)$,

$|\psi'\rangle_{R'A}$ purifies ρ'.

But $I_c(R'\rangle B')_{I\otimes N(|\psi'\rangle\langle\psi'|)}$

$= S(B')_{N(\rho')} - S(E)_{N^c(\rho')}$

$= S(B')_{N(\rho')} - S(E)_{N^c(\rho)}$ $\leftarrow$ use 1st diagram.

$(\neq) \geq S(B)_{N(\rho)} - S(E)_{N^c(\rho)} = Q^{(1)}(N)$ by optimality of $|\psi\rangle$

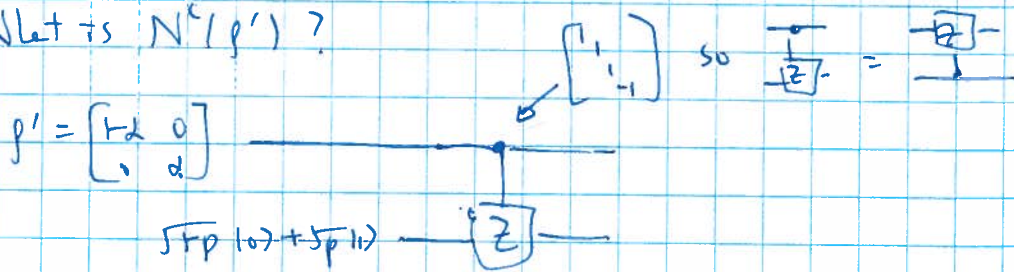
$\therefore$ ρ', which is diagonal, is also optimal.

$\therefore$ optimal state $|\psi'\rangle = \sqrt{1-\alpha}\,|00\rangle + \sqrt{\alpha}\,|11\rangle$.

$$\rho' = \begin{bmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

$N(\rho') = \rho'$ $\quad (\because Z\rho' Z = \rho')$

What is $N^c(\rho')$?

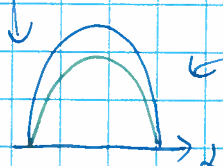


$$N^c(\rho') = (1-\alpha)\begin{bmatrix} 1-p & \sqrt{1-p}\sqrt{p} \\ \sqrt{1-p}\sqrt{p} & p \end{bmatrix} + \alpha\begin{bmatrix} (1-p) & -\sqrt{1-p}\sqrt{p} \\ -\sqrt{1-p}\sqrt{p} & p \end{bmatrix}$$

$$= \begin{bmatrix} 1-p & (1-2\alpha)\sqrt{1-p}\sqrt{p} \\ (1-2\alpha)\sqrt{1-p}\sqrt{p} & p \end{bmatrix}$$

So $\max_{\alpha} \; S\left(\begin{bmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{bmatrix}\right) - S\left(\begin{bmatrix} 1-p & (1-2\alpha)\sqrt{1-p}\sqrt{p} \\ (1-2\alpha)\sqrt{1-p}\sqrt{p} & p \end{bmatrix}\right)$

Attained at $\underbrace{\alpha = \tfrac{1}{2}}_{|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)}$, so $Q(N_p) = Q^{(1)}(N_p) = 1 - h(p)$.



eg. Amplitude damping channel.

You will see in A4 that it is degradable,

again the degrading map is another amplitude damping channel

and you'll find $Q(N_\gamma)$!

Characterization of degradable channels with qubit outputs:

Wolf & Perez-García 0607070

$\forall\ N = B(\mathbb{C}^2) \to B(\mathbb{C}^2)$

$\underbrace{\text{rank}\,(I \otimes N(|\Phi\rangle\langle\Phi|))}_{\text{min \# Kraus ops}} \leq 2 \Rightarrow N$ degradable / anti degradable.

Cubitt, Ruskai, (G) Smith 0802.1360

$\forall\ N = B(\mathbb{C}^d) \to B(\mathbb{C}^2)$

N degradable $\Rightarrow \underbrace{\text{rank}\,(I \otimes N(|\Phi\rangle\langle\Phi|))}$ ≤ 2 & $d \leq 3$.

For larger output dim, a small Choi-rank is not for degradable channels