lec17

Graeme Smith

- Depolarizing channel
  Pauli Channel
  Degenerate codes

$\Big\}$ Lower bounds on capacity.

- upper bounds

- Privacy.

$$N(\underset{\text{qubit}}{\rho}) = (1-p)\rho + \frac{p}{3}\left[X\rho X + Y\rho Y + Z\rho Z\right]$$

$$= (1-q)\rho + q\frac{I}{2}$$

$$\left(1-\frac{4p}{3}\right)\rho + \frac{p}{3}\left(\rho + X\rho X + Y\rho Y + Z\rho Z\right)$$

$$= \left(1-\frac{4p}{3}\right)\rho + \frac{4p}{3}\cdot\frac{I}{2}$$

$$Q^{(1)}(N) = \max_{|\psi\rangle_{RA}} I(R\rangle B)_{I\otimes N(|\psi\rangle\langle\psi|_{RA})}.$$

achievable $\therefore Q(N) \geq Q^{(1)}(N)$

$Q^{(1)}(N) = 1 - H(p) - \log 3$. $\left(\text{achieved @ } |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \right)$

Error ~ $(1-p, \frac{p}{3}, \frac{p}{3}, \frac{p}{3})$

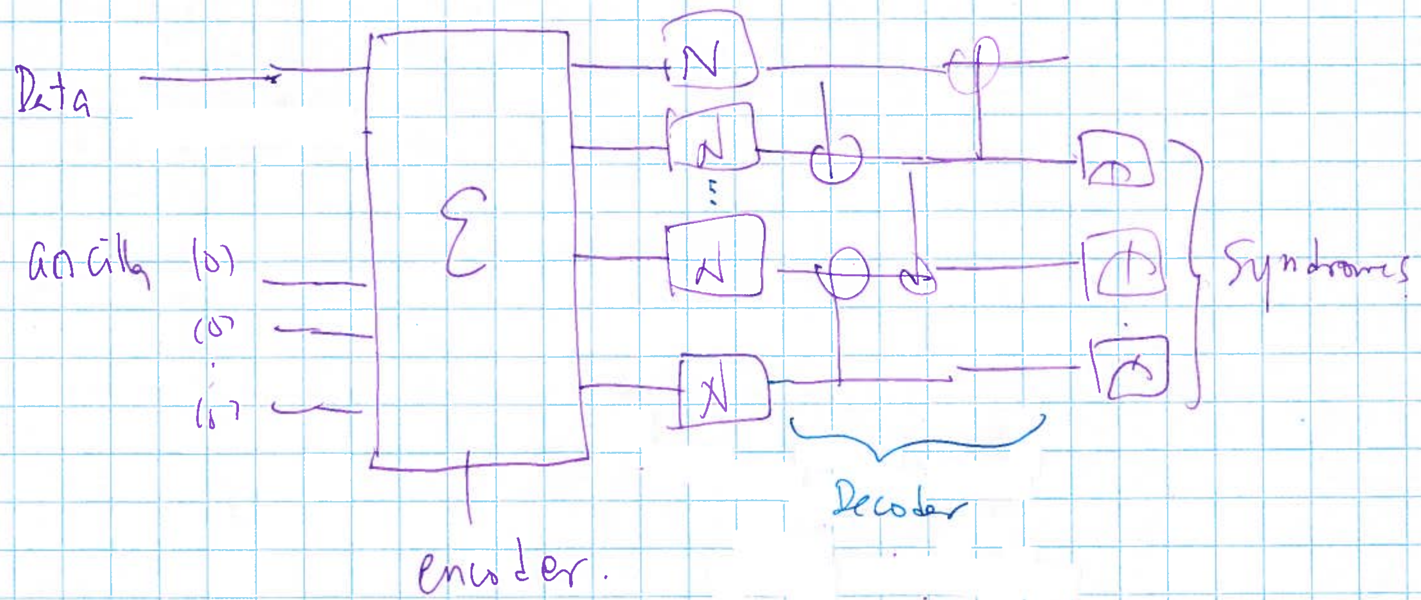
prob for I, X, Y, Z

Given n uses of N, approx $2^{nH(E)}$

$\approx 2^{n(H(p) + p\log 3)}$

# of typical Pauli errors.

Using non-deg codes, $1 - H(E)$ is max achievable rate.



uses of N = # syndrome bits + # data qubits.

Nondeg code $\Rightarrow$

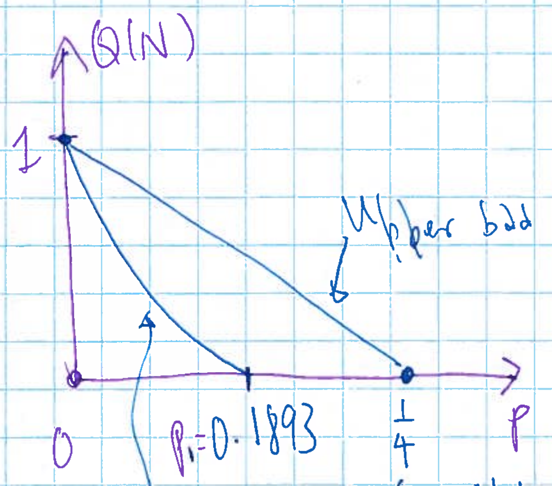
at most 1 error per syndrome.

$\therefore$ Requires $\approx n H(\varepsilon)$ syndrome bits.

$\therefore$ Can fit at most $n(1 - H(\varepsilon))$ data qubits.

$\therefore$ rate $\leq 1 - H(\varepsilon) = Q^{(1)}(N)$.

Now focus on depolarizing channel.



$\leftarrow$ N becomes antidegradable $\therefore Q(N)=0$

$Q^{(1)}(N) = 1 - H(p) - p\log 3$

lower bnd

Note: $Q(N) \geq Q^{(1)}(N)$ in general

but $Q^{(1)}(N^{\otimes n}) \leq Q(N^{\otimes n}) = n\,Q(N)$.

by def of Q.

we will now see this for some N.

$\therefore\ Q(N) \geq \frac{1}{n} Q^{(1)}(N^{\otimes n}) > Q^{(1)}(N)$

Want

$|\phi\rangle_{A A_1 \cdots A_n}$

NB $A = R$ in earlier lectures

$A_1 \cdots A_n$ inputs.

To max

$S(B) - S(E)$ evaluated on $I \otimes U_N^{\otimes n}(|\phi\rangle\langle\phi|)$

iso ext for N

try to suppress $S(E)$.

$$|\phi^+\rangle_{AA'} \xrightarrow{\text{encode } A' \text{ into QECC}} |\phi^+\rangle_{A A_1 \cdots A_n}$$

Pick an interesting block length.

$|0\rangle \to |0\rangle^{\otimes 5}$

$|1\rangle \to |1\rangle^{\otimes 5}$

Stabilizer generators:

$$\left.\begin{matrix} Z & Z & 1 & 1 & 1 \\ 1 & Z & Z & 1 & 1 \\ 1 & 1 & Z & Z & 1 \\ 1 & 1 & 1 & Z & Z \end{matrix}\right\} \text{4 generators}$$

$\boxed{Z\ 1\ 1\ 1\ Z}$ as well?

It's already generated.

Why interested in this code?

To reduce the entropy of $\mathcal{E}$.

Not a coincidence $p_1 \approx \frac{1}{5}$.

logical state

Let the encoded state be $|\psi_L\rangle$ in the 5-bit QECC.

How does an error act on $|\psi_L\rangle$?

$$\left(|\psi\rangle = a|0\rangle + b|1\rangle \rightarrow |\psi_L\rangle = a|0\rangle^{\otimes 5} + b|1\rangle^{\otimes 5} \right)$$

Say what is $Z1111|\psi_L\rangle$?

"+" → "−".

$$|+_L\rangle \rightarrow |-_L\rangle$$

logical.

$$|-_L\rangle \rightarrow |+_L\rangle.$$

$\therefore Z1111$ acts as a logical Z.

But Z on any qubit acts as a logical Z.

This code works well to correct for bit flip error.

eg.

$\alpha|00000\rangle + \beta|11111\rangle$

$\rightarrow \alpha|\underline{\underline{1}}0000\rangle + \beta|\underline{\underline{0}}1111\rangle$. (easy to correct).

Intuition: good for X errors.

collapses diff Z errors as one.

We can substantiate this intuition with a careful calculation (see lec 13 S2010 notes)

$$|\phi_L\rangle_{A A_1 A_2 . A_5} \xrightarrow[\text{Iso ext}]{U_N^{\otimes 5}} |\rho\rangle_{AB_1 \ldots B_5 E_1 \ldots E_5}.$$

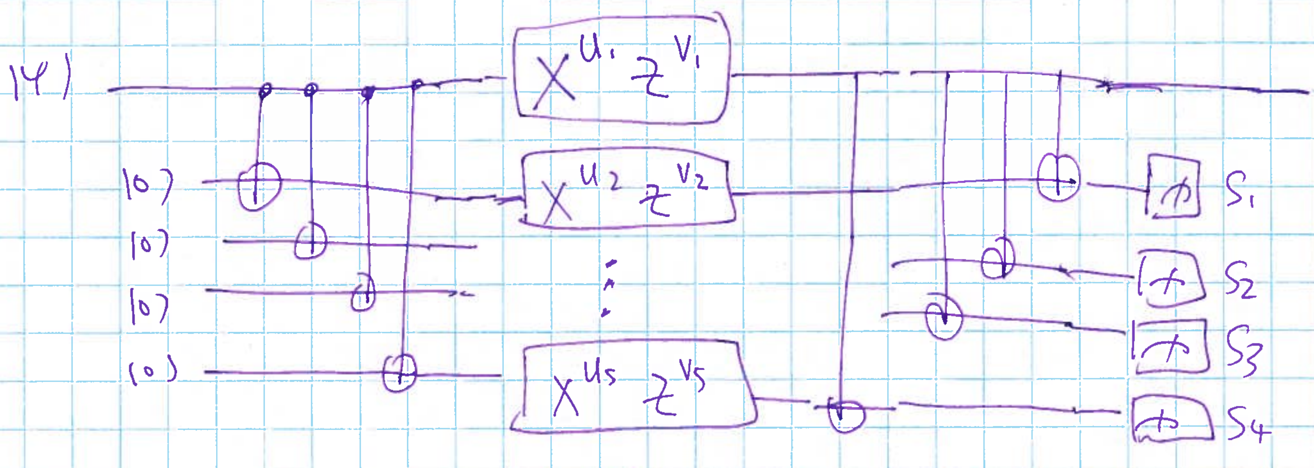
$$I(A \rangle B^5)_{\rho_{AB^5}} = S(B^5) - S(E^5).$$

$(\mathrm{Tr}_{E_1 E_2 \ldots E_5} |\rho\rangle\langle\rho|)$

$S(B^5)$ goes down a little.

$S(E^5)$ goes down much.

What the code does:



One kraus operator from $N^{\otimes 5}$

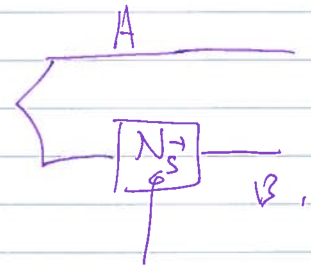
Decoding may decrease $I(A\rangle B^5)_{\rho_{AB^5}}$ but reduces the Q data dim so easier to analysis.

After decoding:

$$I(A\rangle B_1 S_1 S_2 S_3 S_4)$$

$$= \sum_{\vec{s} = s_1 s_2 s_3 s_4} p_{\vec{s}}\, I(A\rangle B)_{\rho^{\vec{s}}_{AB_1}}$$

The decoded state on AB_1 is like one coming from



a random Pauli channel with p_I, p_X, p_Y, p_Z depending on $\vec{s}$.

One can also calculate $p_{\vec{s}}$.

Together, can show

$$\frac{1}{5} Q^{(5)}(N_p^{\otimes 5}) > 0 \quad \text{for } p < 0.1904$$

or even $p < 0.19088$ with better codes.

Why so difficult?

Even for $N^{\otimes 5}$, numerical search for $Q^{(1)}(N^{\otimes 5})$ tends to get stuck in ~~metastat~~ local optimum.

Where does $p=\frac{1}{4}$ special point coming from?

- $Q^{(1)}(N) = Q(N)$ if degradable.
- $I^{coh}(N, \phi)$

$$\boxed{|\phi_{AA'}\rangle \xrightarrow{I_A \otimes N} \rho_{AB}} \quad I_c(A\rangle B)_{\rho_{AB}}$$

Note: $I_c(A\rangle B)_{\rho_{AB}}$.

$$\| $$

$$S(N(\phi_{A'})) - S(\tilde{N}(\phi_{A'}))$$

complementary channel.

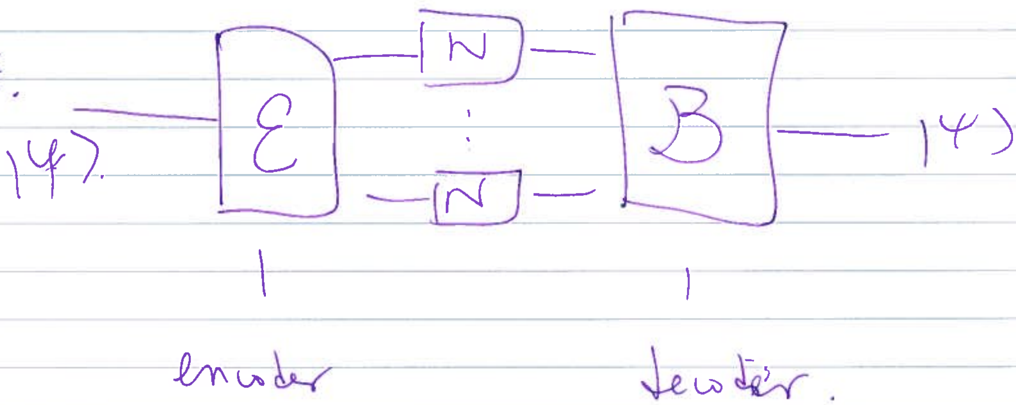
For degradable channel.

$I_c(A\rangle B)_{\rho_{AB}}$ convex wrt $\phi_{A'}$.

If $N = D \circ M$.

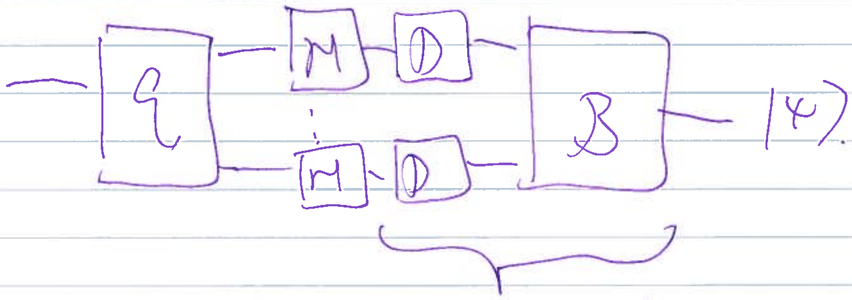
then $Q(N) \leq Q(M)$

Proof:



Say above code has rate R.

Now



call this the new decoder B'

So obtain a code at rate R for M.

Thus $Q(N) \le Q(M) = Q^{(1)}(M)$.

degradability of M

$$N_p(\rho) = (1-p)\rho + \left[\frac{p}{3} X\rho X + \frac{p}{3} Y\rho Y + \frac{p}{3} Z\rho Z\right]$$

~~Examp~~

Degradable channels:

- Amplitude damping $\gamma \le \frac{1}{2}$
- bit flipping, phase flip, Y-flip.

AD channel gives strong upper bdd

but 2nd set of channels are easier to understand.

Let $$M(\rho) = \frac{1}{3}\Big[N_p^X(\rho) \otimes |0\rangle\langle 0|_F + N_p^Y(\rho) \otimes |1\rangle\langle 1|_F + N_p^Z(\rho) \otimes |2\rangle\langle 2|_F\Big]$$

$(1-p)\rho + pX\rho X$

$$N = \underbrace{\mathrm{Tr}_F}_{D.} M.$$

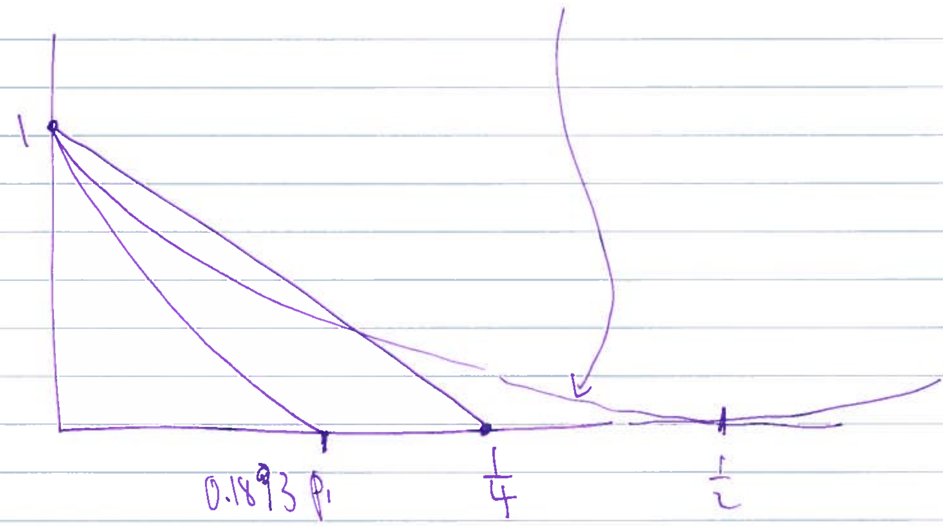
CENTRE DE RECHERCHES MATHÉMATIQUES

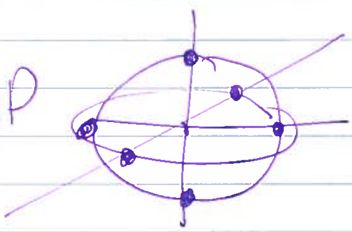
Fact: $\sum_x p_x N_x \otimes |x\rangle\langle x|_F$ is degradable

if each of N_x is.

Now use convexity to show

$Q^{(1)}(M) = 1 - H(p)$



Depolarizing channel = AP  to 6 fix points.

Note degenerate cases show that

$$Q^{(1)}(N \otimes M) \neq Q^{(1)}(N) + Q^{(1)}(M)$$

in general.

In fact "$\neq$" for $N = M$.

Now derive a diff form of superadditivity:

Consider "assisted" capacity.

A symmetric channel A is degradable & anti degradable.

Also $Q(A) = 0$.

Suppose A is given ~~sender & receiver~~ for free.

If you don't use the free channel,
no harm done.

i.e. $Q(\mathcal{N}) \leq Q(\mathcal{N} \otimes \mathcal{A}) \leq \max_{\mathcal{A}} Q(\mathcal{N} \otimes \mathcal{A})$

$$= \max_{\mathcal{A}} \left[\lim_{n \to \infty} \frac{1}{n} Q^{(1)} \left(\mathcal{N}^{\otimes n} \otimes \mathcal{A}^{\otimes n} \right) \right]$$

$$= \max_{\mathcal{A}} Q^{(1)} (\mathcal{N} \otimes \mathcal{A}) \; !$$

Define $|(i,j)\rangle \xrightarrow{V} \begin{cases} \frac{|ij\rangle + |ji\rangle}{\sqrt{2}} & i \neq j. \\ |ij\rangle & i = j. \end{cases}$ $\quad i \leq j$

$$\mathcal{A}(\rho) = \mathrm{Tr}_2 V \rho V^{\dagger}.$$

symmetric.

CENTRE
DE RECHERCHES
MATHÉMATIQUES

$$Q(\mathcal{N}) \leq Q(\mathcal{N} \otimes \mathcal{A}) = Q^{(1)}(\mathcal{N} \otimes \mathcal{A})$$

the channel stated early.
with ∞ dim.

The RHS cannot be evaluated in general

but if $\mathcal{N}$ degradable, it can be.