Degradable & anti degradable channels:

Def: N is called degrable if $\exists\, D$ (TCP map) — sometimes called the degrading map.

s.t. $D \circ N = N^c$

Def: N is called anti degrable if N^c is degradable.

Intuition: if N is degradable, "Bob is better than Eve".

so . . . anti . . . . worse . . . .

NB: The decoding map D exchanges N & N^c.

Def: N is called "symmetric" if N is both degrable & anti degradable.

eg. Let N_p = erasure channel with error prob p.

Then N_p is degradable for $p \leq \frac{1}{2}$

- - - - anti deg - - $p \geq \frac{1}{2}$.

Obs $N_p^c = N_{1-p}$ (use iso ext shown earlier)

Obs: $N_q \circ N_p = N_{p+q-pq}$ (Here N_q has input space spanned by $|0\rangle, |1\rangle, |2\rangle$.)

Pf: $N_q \circ N_p(\rho)$

$= N_q((1-p)\rho + p|2\rangle\langle 2|)$

$= (1-q)[(1-p)\rho + p|2\rangle\langle 2|] + q|2\rangle\langle 2|$.

$= (1-q)(1-p)\rho + (p+q-pq)|2\rangle\langle 2|$.

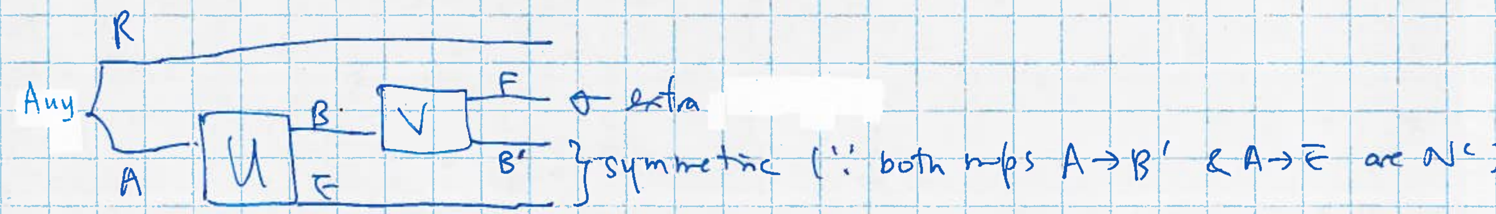
Cor: If $p \leq \frac{1}{2}$, let $q = \frac{1-2p}{1-p}$ nonneg if $p \geq \frac{1}{2}$

then $p+q-pq = p + q(1-p)$

$= p + 1 - 2p = 1 - p$.

then $N_q \circ N_p = N_{1-p}$ ∴ N_p degradable.

Another characterization of degradable channels:



where V = isometric extension of the degrading map

NB: V transmits F from Bob to Eve.

NB: F belongs to Eve for anti deg channels.

NB: F trivial if $\mathcal{N}$ symmetric

<u>Thm</u> If N antidegradable, then $Q(N) = 0$.

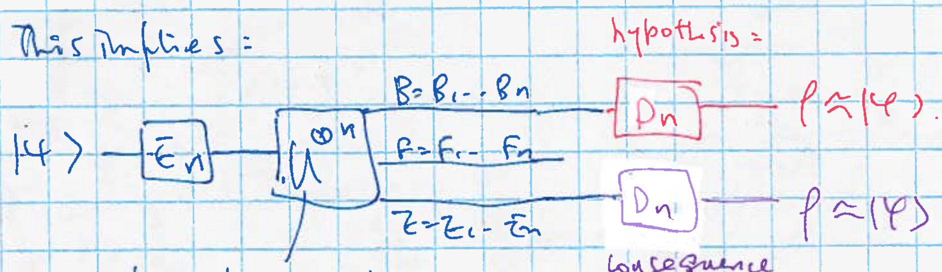
<u>Thm'</u> In fact, not a single qubit can be sent with arbitrarily large number of uses of N.

"<u>Pf:</u>" If Alice can send q data to Bob, Eve can get a copy too, implying cloning.

<u>Pf:</u> If $|\psi\rangle \in \mathbb{C}^d$ can be transmitted with high fidelity, with n uses of N,

$\exists$ E_n, D_n s.t. $|\psi\rangle \to \boxed{E_n} - \boxed{N^{\otimes n}} - \boxed{D_n} \to \rho \approx |\psi\rangle$.

This implies:



isometric extension of N that incorporates the degrading isometry for Eve.

Thus cloning with high fidelity – a contradiction.

Example: Classical channel is symmetric (cf A1)

Example: $Q(N_p) = 0$ $\forall p \geq \frac{1}{2}$ in erasure channels.

Thm [Devetak & Shor]:

If N_1, N_2 degradable, then $Q^{(1)}(N_1 \otimes N_2) = Q^{(1)}(N_1) + Q^{(1)}(N_2)$.

Pf: [$\geq$]

Lemma: Let $\rho_{RB} = \sigma_{R_1B_1} \otimes \varsigma_{R_2B_2}$, $R = R_1R_2$, $B = B_1B_2$

then $I_c(R\rangle B)_\rho = I_c(R_1\rangle B_1)_\sigma + I_c(R_2\rangle B_2)_\varsigma$

Pf (Straight forward) (Omitting states since we only take reduced states).

$I_c(R\rangle B) = S(B) - S(RB)$

$= S(B_1B_2) - S(R_1R_2B_1B_2)$

$\overset{\text{product state}}{=} S(B_1) + S(B_2) - S(R_1B_1) - S(R_2B_2)$

$= I_c(R_1\rangle B_1) + I_c(R_2\rangle B_2)$

Back to pf [$\geq$]:

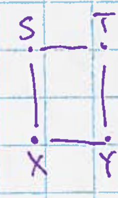
Let optimal input for N_1 be ψ_1
" " " N_2 " ψ_2.

Then $Q^{(1)}(N_1 \otimes N_2) \geq I_c(R_1R_2\rangle B_1B_2)_{\underbrace{I_{R_1R_2} \otimes N_1 \otimes N_2 (\psi_{1\,R_1A_1} \otimes \psi_{2\,R_2A_2})}_{\text{product state}}}$

$\overset{\text{Lemma}}{=} I_c(R_1\rangle B_1)_{I_{R_1} \otimes N_1 (\psi_{1\,R_1A_1})} + I_c(R_2\rangle B_2)_{I_{R_2} \otimes N_2 (\psi_{2\,R_2A_2})}$

$\overset{\text{optimal inputs}}{=} Q^{(1)}(N_1) + Q^{(1)}(N_2)$. //

Pf [$\leq$]



Lemma: For any state in 4 systems $STXY$

$$S(ST|XY) \leq S(S|X) + S(T|Y)$$

Pf: RHS − LHS

$$= S(S|X) + S(T|Y) - S(ST|XY)$$

$$= S(SX) - S(X) + S(TY) - S(Y) - [S(STXY) - S(XY)]$$

$$= S(SX:TY) - S(X:Y)$$

≥ 0 by monotonicity by QMI

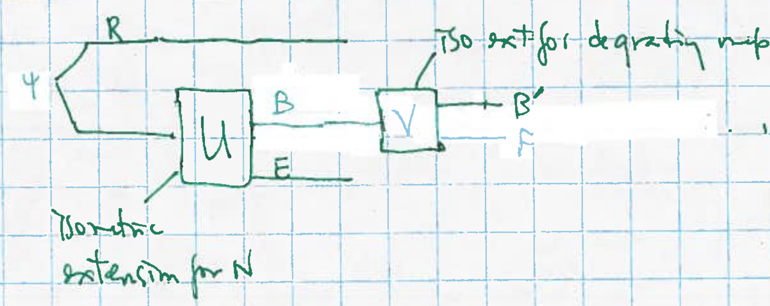
Lemma: for any degradable channel & any input ψ_{RA}

$$I_c(R\rangle B)_{I\otimes N(\psi)} = S(F|E)$$

$$\mathrm{tr}_{RB'}\left((V\otimes I_{RE})(U\otimes I_R)(\psi)(U^\dagger\otimes I_R)(V^\dagger\otimes I_{RE}) \right)$$

σ

where



, B', E sym.

Pf: $I_c(R\rangle B)_{I\otimes N(\psi)}$

$$= S(B) - S(E)$$

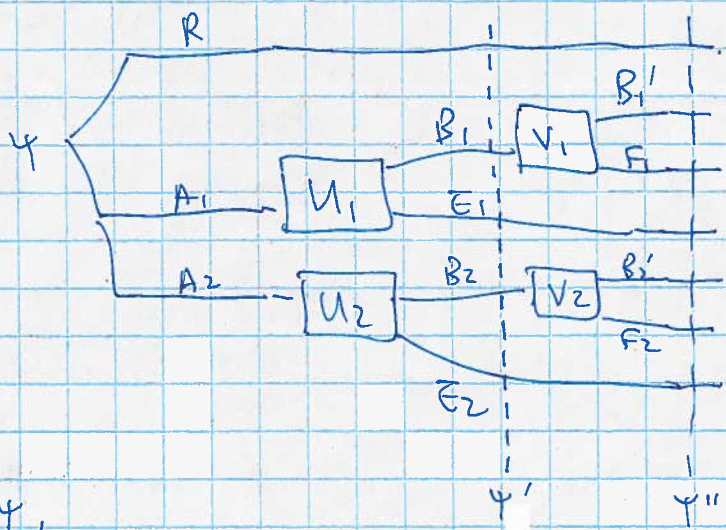
V iso $= S(B'F) - S(E)$

degradable B', E sym $= S(EF) - S(E)$

$$= S(F|E)_\sigma$$

Back to pf [$\le$].

Suppose U_1, V_1 are iso ext to N_1 & its degrading map
U_2 V_2 - - - N_2 - - .
Then $U_1 \otimes U_2$, $V_1 \otimes V_2$ - - - - - $N_1 \otimes N_2$ - - - - - .



($B_1'B_2'$ sym E_1E_2)

$\therefore \forall \psi$,

$I_c(R > B_1B_2)_{\psi'}$

Lemma
$= S(F_1F_2 | E_1E_2)_{\psi''}$

Lemma
$\le S(F_1|E_1)_{\psi''} + S(F_2|E_2)_{\psi''}$

$= I_c(\underbrace{RB_2E_2}_{R_1} > B_1)_{\psi'} + I_c(\underbrace{RB_1E_1}_{R_2} > B_2)_{\psi'}$.

$\le Q^{(1)}(N_1) + Q^{(1)}(N_2)$

$\therefore Q^{(1)}(N_1 \otimes N_2) \le Q^{(1)}(N_1) + Q^{(1)}(N_2)$

Thm If $\mathcal{N}$ degradable

then $Q^{(r)}(\mathcal{N}) = Q^{(1)}(\mathcal{N}) \quad \forall r \in \mathbb{Z}^+$

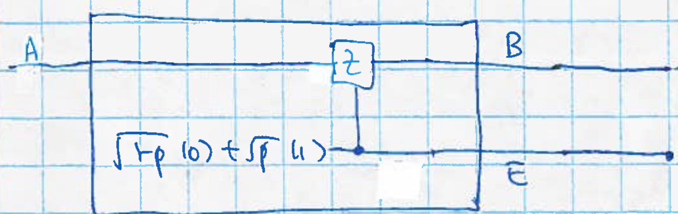
In particular, $Q(\mathcal{N}) = Q^{(1)}(\mathcal{N})$.

Cor. $Q(\mathcal{N}_p) = 1-2p$ for erasure channel $\mathcal{N}_p$

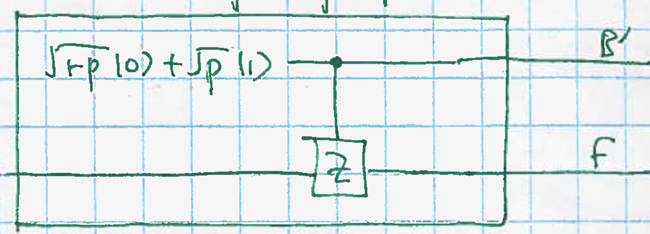
eg. Phase damping channel.

$N_p(\rho) = (1-p)\rho + p Z \rho Z^\dagger$

A possible isometric extension:



and a degrading map:



Note: the degrading map commutes with N_p
so B' & E are symmetric.

$\therefore N_p$ is degradable $\forall p \in [0,1]$. (worse case is $p = \frac{1}{2}$)

Now let any $|\psi\rangle_{RA}$ to be any input,

and $|\alpha\rangle_{RBE} = \sqrt{1-p}\,|0\rangle_E\,|\psi\rangle_{RB} + \sqrt{p}\,|1\rangle_E\,(I \otimes Z)\,|\psi\rangle_{RB}$

be the output. From numerics. optimal $|\psi\rangle_{RA} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

$I_c(R\rangle B)_\alpha = S(B) - S(E) = 1 - H(p)$

~~$\underbrace{\leq 1 \quad \leq H(p)?}_{\text{achievable}}$ [$S(E) < H(p)$ if $|\psi\rangle_{RB}$ not ortho to $(I\otimes Z)|\psi\rangle_{RB}$]~~

~~if $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$~~

$\therefore Q(N) = Q^{(1)}(N) = 1 - H(p)$

eg. Amplitude damping channel.

You will see in A6 that it is degradable,

again the degrading map is another amplitude damping channel

and you'll find $Q(N_\gamma)$!

Characterization of degradable channels with qubit outputs:

Wolf & Perez-García 0607070

$\forall\, N : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$

rank $(I \otimes N(|\Phi\rangle\langle\Phi|)) \leq 2 \Rightarrow N$ degradable / anti degradable.

↑ min # Kraus ops

Cubitt, Ruskai, (G) Smith 0802.1360

$\forall\, N : B(\mathbb{C}^d) \to B(\mathbb{C}^2)$

N degradable $\Rightarrow \underbrace{\text{rank}(I \otimes N(|\Phi\rangle\langle\Phi|))}$ ≤ 2 & $d \leq 3$.

For larger output dim, a small Choi-rank is not for degradable channels

• Consider the random Pauli channel:

$$\mathcal{N}_{\vec{q}}(\rho) = q_0\,\rho + q_1\,X\rho X + q_2\,Y\rho Y + q_3\,Z\rho Z$$

where $0 \le q_i$, $\sum_{i=0}^{3} q_i = 1$ and WLOG, $q_0 = \max_i q_i$

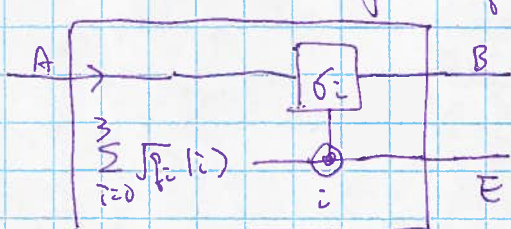
• A special case is the depolarizing channel:

$$\mathcal{N}_q(\rho) = (1-q)\,\rho + \frac{q}{3}\left[X\rho X + Y\rho Y + Z\rho Z\right] \qquad \text{for } q \in [0, \tfrac{3}{4}]$$

$$= \left(1 - \frac{4q}{3}\right)\rho + \underbrace{4\frac{q}{3}}_{p}\ \underbrace{\left[\rho + X\rho X + Y\rho Y + Z\rho Z\right]\frac{1}{4}}_{\frac{I}{2}}$$

$$= (1-p)\,\rho + p\,\frac{I}{2} \qquad \text{where } p = \frac{4q}{3}$$

• An isometric extension for $\mathcal{N}_{\vec{q}}$ is given by:



• Choi-rank = 4 $\therefore$ not degradable.

• Claim: $Q^{(1)}(\mathcal{N}_{\vec{q}}) := \max_{|\psi\rangle_{RA}} I_c(R\rangle B)_{I\otimes \mathcal{N}_{\vec{q}}(|\psi\rangle\langle\psi|)}$

$$= \max\{1 - H(\vec{q}),\ 0\}.$$

In particular, optimal input is $|\psi\rangle_{RA} = \begin{cases} |\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) & \text{if } 1 - H(\vec{q}) \ge 0 \\ |0\rangle|0\rangle & \text{otherwise.} \end{cases}$

Proof: extensive numerics (+ continuity of $I_c(R\rangle B)_{I\otimes\mathcal{N}_{\vec{q}}(|\psi\rangle\langle\psi|)}$ as a function of $|\psi\rangle$).

In A5, you showed $Q^{(n)}_{\vec{f}}(N) \geq \underbrace{1 - H(\vec{f})}_{\text{this happens to be } Q^{(1)}(N_{\vec{f}})}$ for n large enough.

It turns out $Q^{(n)}(N_{\vec{f}}) \gneq 1 - H(\vec{f})$ for some $N_{\vec{f}}$, due to degenerate codes.