Last time

H 5 16. Thm.

\[ C(n) = \sup_r \chi^{(r)}(n) \]

where \[ \chi^{(r)}(n) = \max_{x \in F} \chi(x^r, N(fx)^r) \]

Consequences:

1. \[ C(n) = \sup_r \chi^{(r)}(n) \geq \chi^{(1)}(n) \geq \chi^{(0)}(n) \]

2. \[ C(n) = 0 \Rightarrow \chi^{(1)}(n) = 0 \Rightarrow N \text{ "null" claim} \]

3. \[ C(n) \text{cts in } N \text{ for the metric derived from the Schwartz norm} \]

4. How to find \( \chi^{(r)}(n) \)?
Def: In the expression \( X(N) := \max_{\{\rho_x, N(\rho_x)\}} X(x_{\rho_x}) \),
\( \{\rho_x, \rho_x^2\} \) is called the "optimal ensemble" for \( N \) if it achieves the max.

Properties of the optimal ensemble:

Ulmann 970114, Schumacher & Westmoreland 991222

1. \( \rho_x \)'s can be chosen pure
2. \( d^2 \) pure states are sufficient, where \( d \) = output dimension of \( N \)

Pf: A3 (similar to [Ref.])
eq. 1 \[ N_p = d - \text{dim depolarizing channel} \]

\[ N_p(f) = (1 - p) p + p \frac{d}{d} \]

\[ X^{(1)}(N_p) = \max_{p_k \geq k} \left( \sum_k p_k N(N_{1k} < Y_{1k}) \right) - \sum_k p_k \log \left( \frac{1}{N_{1k}} \right) \]

For the 2nd term:

\[ N(1_{1k} < Y_{1k}) = (1 - p) 1_{1k} X_{1k} + p \frac{d}{d} \]

Spectrum: \[ (1 - p + \frac{d}{d}, \frac{d}{d}, \ldots, \frac{d}{d}) \]

\[ \text{independence of } X_{1k} \]

\[ \left( \frac{d - 1}{d} \right) \times \text{times} \]

\[ S(N(1_{1k} < Y_{1k})) = -(1 - p + \frac{d}{d}) \log(1 - p + \frac{d}{d}) - \frac{d - 1}{d} p \log \frac{d}{d} = s \]

\[ X^{(1)}(N_p) = \max_{p_k \geq k} \left( \sum_k p_k N(N_{1k} < Y_{1k}) \right) - s \]

attained when \[ \sum_k p_k N(N_{1k} < Y_{1k}) = \frac{d}{d} \]

when \[ p_k = \frac{1}{2}, N_{1k} = \infty \]

\[ = \log d - s. \]

For \( d = 2 \), \[ X^{(1)}(N_p) = 1 - h\left(\frac{1}{2}\right) \] (we will see this is \( C(N_p) \))

Capacity of classical ESC with hybrid prob. \( \frac{d}{d} \).

What is the code for achieving this? (d-dimensional)

Draw \( 2^n \) code words from all possible \( n \)-tuples of symbols in \( \{0, 1, \ldots, d-1\} \) and restrict to strongly-typical sequences.

Due to overlaps of the set bits, expect \( T \) entanglement & so is the deciding measurement.
In general, if 

1. $N$ is unitary \((N(N) = I)\)

2. Each \(y_i\) minimizes \(S(N(y_i) < y_i) = 5\)

3. \(\frac{1}{d} \in \mathcal{C}(y_i)\)

then \(\{p_i; y_i\}\) is optimal ensemble and \(X^{(i)}(N) = \log d - 5\)

where \(\sum_{i} p_i y_i = \mu\).

Ex: find \(X^{(i)}(N)\) for \(N(f) = 0.8p + 0.15 xp + 0.05 \geq p^2\)

Ex: find \(X^{(y)}(E_p)\) for \(E_p(f) = (1-p) + p I(X = E)\)

\[\text{Prob} \\quad \text{Erase} \quad \text{Channel} \quad \text{Dim} \quad \text{Error state ortho to input space}\]

Compare \(X(E_p)\) to the classical analogue.

Note: \(E_p\) not unitary.
Properties of \( \mathbf{\Omega} \):

1. The most distinguishable inputs need not be orthogonal.

2. For the case \( \mathbf{\Omega} \) is non-orthogonal, \( \mathbf{\Phi} \) may be needed.

- Frieder PRL 79 (1997) 1162 found the first channel that requires non-orthogonal optimal \( \mathbf{\Omega} \).

- Amplitude damping channel shares the same property (99(2)2).

Amplitude damping channels

\[
N(\phi) = A_0 \mathbf{\Phi} A_0^* + A_i \mathbf{\Phi} A_i^* 
\]

\( A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\phi} \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & \sqrt{\phi} \\ 0 & 0 \end{bmatrix} \)

\( \begin{align*}
A_0 \rightarrow (a_0 + \delta a_1) \\
A_i \rightarrow \sqrt{\phi} b_1 i a_0 
\end{align*} \)

If \( |a_0, b_1\rangle \) are energy eigenstates, then \( A_i \rightarrow \sqrt{\phi} b_1 a_0 \)

\( \begin{align*}
|a_0, b_1\rangle & \rightarrow \left( a_0 + \sqrt{\phi} b_1 i a_0 \right) \\
& = a_0 + b_1 i a_0 \\
& = (1 + i\sqrt{\phi}) a_0
\end{align*} \)

This AO represents de-excitation.

- Restrict to \( 140^\circ \) \( \mathbf{\Omega} \langle \mathbf{\Omega} \rangle \),

\[
\max @ p_0 = 0.5, \quad \chi = 0.4562.
\]

- Instead, \( \chi = 0.4717 \)

for some \( 140^\circ \), \( 141^\circ \)

with \( \langle 45^\circ | 41^\circ \rangle \approx \cos 80^\circ \).
How hard is it to calculate or estimate $\chi^{(n)}(N)$?

Beigi & Shor 0707.2090:

Let $c \in \mathbb{R}$. To decide whether $\chi^{(n)}(N) > c$ or $c - \varepsilon < \ldots$

for $\varepsilon = \frac{1}{\text{poly}(d)}$ ($d = \text{input dim}$) is NP complete.

NB: The above does not imply hardness of estimating $\chi(N^n)$. / more structure
(4) $x^{(r)}(N) \text{ vs } x^{(1)}(N)$

**Def:** $x$ strongly additive on $N$

\[ \text{if } N', N'' = x^{(r)}(N) + x^{(s)}(N') \]

**Def:** $x$ weakly additive on $N$

\[ \text{if } x^{(1)}(N) = x^{(r)}(N) \]

**NB:** $\forall N, N', x^{(r)}(N \odot N') \geq x^{(r)}(N) + x^{(s)}(N')$

**Pf:** let $\{p_x, p_y\}$ and $\{q_x, q_y\}$ be the product ensembles for $N \& N'$ respectively.

Then $x^{(r)}(N \odot N')$

\[ \geq x\left(\{p_x \odot q_y, N \odot N'\}(p_x \odot q_y)\right) \]

\[ = S\left(\{XY : BB'\}\right) \]

\[ \lambda = \sum_{x,y} p_x q_y \left(x \odot (x \odot y) \odot N(b_x) \otimes N'(b_y)\right) \]

\[ = \sum_{x \odot y} \left(\sum_{x} p_x (2x \odot y) \odot N(b_x) \otimes N'(b_y)\right) \]

\[ = S(BB') - S(\{BB', 1, xy\}) \]

\[ = S(B) + S(B') - \sum_{xy} p_x q_y S\left(N(b_x) \otimes N'(b_y)\right) \]

\[ \leq S(N(b_x)) + S(N'(b_y)) \]

\[ = S(N(b_x)) + S(N'(b_y)) \]

\[ \geq \sum_{x \in B} S(N(b_x)) + \sum_{y \in N'} S(N'(b_y)) \]

\[ = x^{(1)}(N) + x^{(1)}(N') \]
Thus additive $\iff$ optimal ensemble for $N \otimes N'$ is
the product of optimal ensembles for $N$ & $N'$
$\iff$ optimal ensemble contains product states only

NB: Strongly additive $\implies$ weakly additive.

Known additivity result:

(i) If $N$ is entanglement breaking, then $\chi$ is
strongly additive on $N \otimes N(030203)$
(w.r.t. $(|N\rangle = \chi^{(1)}|N\rangle$).

Def: $N$ is entanglement breaking
if $\forall \rho_{RA}$, $I \otimes N(\rho_{RA})$ is separable (convex combination of product states)

\[
\begin{tikzcd}
\rho_{RA} \ar[bend left=35]{r}{R} \ar{r}{A} \ar{d}{N} \ar{r}{B} & \text{Separable}
\end{tikzcd}
\]

Characterization: $N$ entanglement breaking $\iff$ $N$ = CCA channel

ie $\chi$:

\[
\begin{tikzcd}
\chi \ar[bend right=35]{}{\chi} \ar{r}{\rho_{MN}} \ar{d}{N} \ar{r}{C} \ar{r}{A} \ar{d}{\chi} & \text{CCA}
\end{tikzcd}
\]

Proof idea: $N$ ent breaking $\iff$ $\chi(N)$ separable $\iff$ $N$ is CCA.

NB: Classical channels & W-boxes are entanglement breaking

Insight: Entangled ensembles & entangled output (from diff uses of $N$)
needed for sup. additivity.
(2) If $N$ is unitary (i.e., $NUU^* = I$) and $N$ has 2-dim input & output, then $X$ is strongly additive on $N$.

(King 0103156)

Cor: $(N) = X^U(N)$ for qubit unitary channels

Aside: all qubit unitary channels have the form:

$$\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$$

where $U, V \in U(2)$, 

$$\xi(p) = \frac{1}{3}p: \xi_1: \xi_2: \xi_3:$$

Random Pauli channel

(3) If $N$ is the $d$-dim depolarizing channel then $X$ is strongly additive on $N$.

Cor: $(N) = X^U(N)$.

Unknown additivity:
Most channels, including the amplitude damping channel $N_D$ so $C(N_D)$ unknown...

Non-additivity:

$$\exists N_1, N_2 \text{ s.t. } X^U(N_1 \otimes N_2) \neq X^U(N_1) + X^U(N_2)$$

Hastings 0809.3972
Def: Entanglement of formation

\[ E_f(\rho) = \min_{\sum \pi_i} \sum \pi_i E(\pi_i) \]

\[ \text{given } \pi_i, \text{ average to be minimized over } \pi_i \text{ under the above constraint} \]

\[ \text{ie } E_f(\rho) = \min \text{ are entanglement in a convex decom of } \rho \text{ into pure states.} \]

Intuition: to make \( f \) un for (arbitrary)

1. For each \( i \), Alice & Bob create \( n \pi_i \) copies of \( \rho_i \)
   using \( n \pi_i E(\rho_i) \) ebits & \( 10 n \pi_i \) qubits.
2. Alice measures \( i \) up \( \rho_i \), repeats in times \( n \pi_i \) to obtain \( \tilde{i}, \tilde{i}_2 \ldots \tilde{i}_n \) and shares outcome with Bob.
3. They place a copy of \( \rho_{\tilde{i}_k} \) in A&B,
   \[ \rho_{\tilde{i}_k} \rightarrow A \tilde{B}_k \]
   \[ \vdots \]
   \[ \rho_{\tilde{i}_n} \rightarrow A \tilde{B}_n \]

Tracing out the states holding \( \tilde{i}, \tilde{i}_2 \ldots \tilde{i}_n \), they obtain \( f_{\tilde{i}} \)

Thus the "entanglement of formation" BDSW '96, where
min over convex decom of \( f \) "minimizes" the qubit cost.
On: Is $\text{Ef}(\rho)$ the min asymptotic & edits needed for copy of $\rho$?

In 2008, Hayden, Horodecki, and Tarnaki defined the entanglement cost as:

$$E_c(\rho) = \lim_{n \to \infty} \frac{1}{n} \left( \text{qubits needed to produce } \rho^{\otimes n} \text{ with accuracy } \varepsilon_n \text{ & w/ free qubits} \right)$$

and proved:

$$E_c(\rho) = (\lim_{n \to \infty} \frac{1}{n} \text{ Ef}(\rho^{\otimes n})) \leq \text{Ef}(\rho)$$

$$\rho^{\otimes n} = \sum_j \frac{1}{d_j} |\phi_j\rangle \otimes |\phi_j\rangle$$

entangled over $A, B, A_2 B_2 - A_B, A_2 B_2$.

gives decomposition better than

$$\rho^{\otimes n} = \left( \sum \rho_j |\psi_j\rangle \langle \psi_j| \right)^{\otimes n}$$

one copy in each of $(A_B, A_2 B_2) - (A_B, A_2 B_2)$.

In 2008, Nor prove TFAE:

1. $\forall N_1, N_2$, $\chi^{(1)}(N_1 \otimes N_2) = \chi^{(1)}(N_1) + \chi^{(2)}(N_2)$

2. $\forall f_1, f_2$, $\text{Ef}(f_1 \otimes f_2) = \text{Ef}(f_1) + \text{Ef}(f_2)$

3. $\forall N_1, N_2,$ $S_{\min}(N_1 \otimes N_2) \geq S_{\min}(N_1) + S_{\min}(N_2)$

[where $S_{\min}(N) = \min_p S(N|p) = \min p$ without entangly]

4. $\forall A, B, A_2 B_2$, $\text{Ef}(\rho_{A1 B_2 A_2 B_2}) \geq \text{Ef}(\rho_{A_1 B_1}) + \text{Ef}(\rho_{A_2 B_2})$

(Lec 8-9, S 2010)
How was $\mathcal{O}$ disposed?

References: Hastings 0809.3972

- Also Brandao, M Horodecki 0907.3210
- Fukuda, Icing, Mosor 0905.3697
- Aubrun, Szarek, CE Werner 0910.1189

Built on failure of $\mathcal{O}$ for Renyi-entropies
- Holevo-Werner CE, Winter, Hayden

Main Ideas:

- Let $\text{input dim} = \text{out put dim} = d$.

- Choose $N_1$: $N_1(f) = \frac{1}{n} \sum_{k=1}^{n} U_k f U_k^\dagger$, $U_k \in U(d)$

  $N_2 = \overline{N}_1$: $N_2(f) = \frac{1}{n} \sum_{k=1}^{n} \overline{U}_k f \overline{U}_k^\dagger$, $\overline{U}_k$ complex conjugate

- Show $\text{Smin}(N_1, \overline{N}_2) \leq 2 \log d - \log n$

Reason:

- Joint input $\langle \overline{\omega} | = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} | j \rangle | j \rangle$ well preserved

Reason:

Transport trick: $A \otimes I (\overline{\omega}) = I \otimes A^T (\overline{\omega})$, $A 
\times p$ as $p' \in \mathbb{C}^n$.

Pf: Let $\langle \overline{\omega} | = 1 \otimes U^\dagger | \chi \rangle$

Thus $U \otimes U \langle \overline{\omega} | = 1 \otimes U^\dagger U \langle \chi \rangle$.
\[ n \text{ out of } n^2 \text{ Kraus ops on } N_1 \otimes N_2 \equiv x \Rightarrow \]

\[ \langle \bar{w} | N_1 \otimes N_2 (| \bar{v} \rangle \langle \bar{v}|) | \bar{w} \rangle \geq \frac{\epsilon}{n}. \]

Worse case spectrum:
\[ \left\{ \frac{1}{n}, \left(\frac{1}{n} \right)^{-1/n^2}, \ldots, \left(\frac{1}{n} \right)^{-2/n^2} \right\}, \] \[ n^2 - 1 \text{ times}. \]

(Note \( N_1 \otimes N_2 (| \bar{v} \rangle \langle \bar{v}|) \text{ rank } n^2. \))

\[ \Rightarrow S_{\text{min}}(N_1 \otimes N_2) \leq S(N_1 \otimes N_2 (| \bar{v} \rangle \langle \bar{v}|)) \]

\[ \leq -\frac{1}{n} \log \frac{1}{n} - \left(1 - \frac{1}{n} \right) \log \left(1 - \frac{1}{n^2}\right) \left(\frac{1}{n^2 - 1}\right) \]

\[ \leq 2 \log n - \frac{\log n}{n}. \]

• Note \( S_{\text{min}}(N_1) = S_{\text{min}}(N_2). \)

Since \( S(N_1 \otimes I) = S(N_2(\bar{v} \langle \bar{v}|)). \)

Difficult part:

\[ \forall \langle \bar{v}, \bar{w} \rangle, \ S(N_1(\bar{w} \langle \bar{w}| \otimes I)) \leq \log n \left(\frac{\text{const.}}{n} + \text{poly}(n) [0, \log 2] \right). \]

Requires choosing \( \bar{w} \)'s according to Haar measure.

Assume \( \exists \text{ input with low entangly out put } \) & get contradiction

Choose \( d \gg n \) gives \( S_{\text{min}}(N_1 \otimes N_2) \leq 2 S_{\text{min}}(N_1). \)

Note large \( d \), non-constructive, variations on \( N_1 \), & proof method.

Tight race \( \approx 2\text{ terms}. \)