"Back to q. capacity:

- Def (coherent info for a q. state)

Let ρ be a state on sys R & B.

$$I_c(R\rangle B)_\rho := \underbrace{[S(B) - S(RB)]}_{-S(R|B)}\rho \quad \text{evaluated on } \rho$$

Aside: why 2 names for the same quantity?

(S(R|B) does not have much "status" until 2005)

e.g. ρ_{RB}	$I_c(R\rangle B) = S(B) - S(RB)$	$S(R:B)$
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$1 = 1 - 0$	2
$\frac{1}{2}(\|00\rangle\langle 00\| + \|11\rangle\langle 11\|)$	$0 = 1 - 1$	1
$\|0\rangle\|0\rangle$	$0 = 0 - 0$	0
$\frac{I}{2} \otimes \|0\rangle\langle 0\|$	$-1 = 0 - 1$	0
$\|0\rangle\langle 0\| \otimes \frac{I}{2}$	$0 = 1 - 1$	0

$I_c(R\rangle B) \approx$ Q correlation (not quite)

$S(R:B) \approx$ total correlation

- Interpretation:

Let E purify RB. (i.e. state $|\psi\rangle_{RBE}$ s.t. $\mathrm{tr}_E |\psi\rangle\langle\psi| = \rho_{RB}$)

Then $I_c(R\rangle B) = S(B) - S(RE)$

$$= S(B) - S(E) \qquad ①$$

Also $S(R:B) - S(R:E) = S(R) + S(B) - S(RB)$

$$- [S(R) + S(E) - S(ER)]$$

$$= S(B) - S(E) - S(E) + S(B) = 2 I_c(R\rangle B). \qquad ②$$

$\therefore I_c(R\rangle B)$ quantifies how much more Bob is correlated with R than Eve is.

NB $I_c(R\rangle B)$ indep of choice of purification.

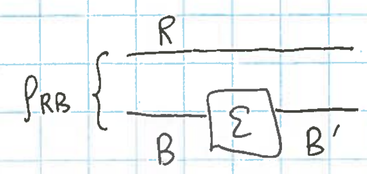
Properties of the coherent information $I_c(R>B) := S(B) - S(RB)$

① Invariant under local unitaries (on R & B separately)

② Invariant under attaching / removing pure states locally.

③ Nonincreasing under local operation on B

Pf:



Note $I_c(R>B) = S(B) - S(RB)$
$= S(B:R) - S(R)$

By monotonicity of QMI, $S(B:R)_\rho \geq S(B:R)_{\underset{R\ \ B}{I\otimes\mathcal{E}(\rho)}}$

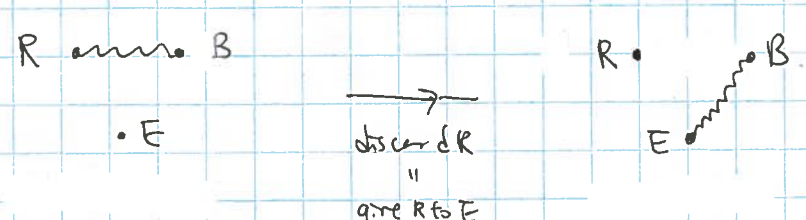
Meanwhile $S(R)$ is evaluated on the same state $\mathrm{tr}_B(\rho) = \mathrm{tr}_{B'}(I\otimes\mathcal{E}(\rho))$ before & after $\mathcal{E}$.

argument for ③ does not apply to ④ ∵ $S(B) \neq S(B')$ in general.

$\therefore\ I_c(R>B)_\rho \geq I_c(R>B')_{I\otimes\mathcal{E}(\rho)}$ ← Q data processing ineq

☆☆ ④ Discarding subsystem of R may increase / decrease $I_c(R>B)$:

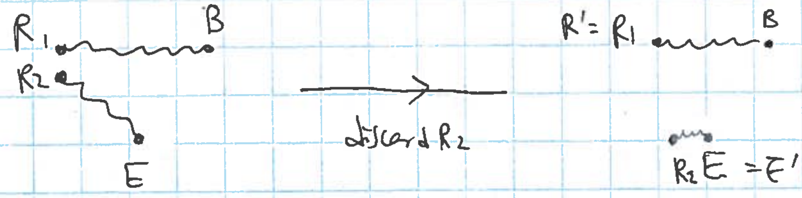
eg 1.



$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{RB}\,|0\rangle_E$ $\qquad$ $|0\rangle_R\,\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{BE}$

$I_c(R>B) = 1$ $\qquad$ $I_c(R>B) = 0$

eg 2



$I_c(R_1R_2\rangle B) = 0$

$\parallel$

$S(B) - S(R_1R_2B) = 1-1$

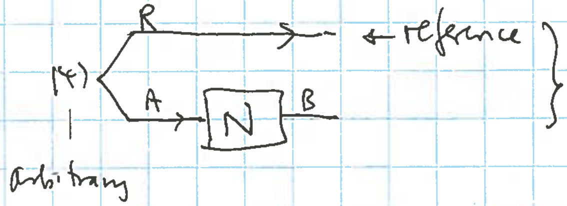
$I_c(R'\rangle B) = 1$

Possible to uncorrelate R from E if subsys of R discarded, I_c φ

Coherent info for channel:

Consider $\xrightarrow{A} \boxed{N} \xrightarrow{B}$

&

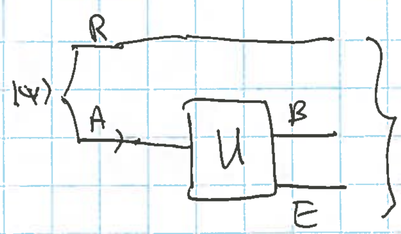


← reference

$I \otimes N(|\psi\rangle\langle\psi|)$

[R ≠ achievable rate here]

Using Stinespring dilation / unitary representation:



$I \otimes U|\psi\rangle = |\phi\rangle_{RBE}$

• Def: $Q^{(1)}(N) = \max_{|\psi\rangle} I_c(R\rangle B)_{I\otimes N(|\psi\rangle\langle\psi|)}$ ← max diff in QMI between RB & RE due to 1 use of N.

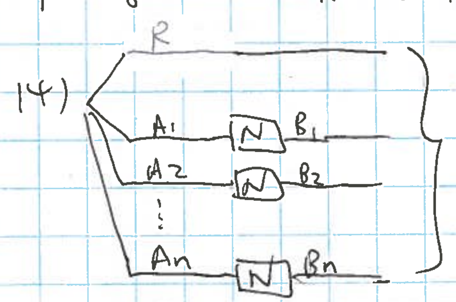
NB: if $|\psi\rangle = |\alpha_1\rangle_R |\alpha_2\rangle_A$, then $S(B) = S(E)$ for $|\phi\rangle_{RBE}$.

$I_c(R\rangle B) = 0$

$\therefore Q^{(1)}(N) \geq 0$. (though I_c can be negative)

• Def: $Q^{(n)}(N) = \frac{1}{n} \underbrace{Q^{(1)}(N^{\otimes n})}$

Optimized over $|\psi\rangle_{RA_1A_2\ldots A_n}$, evaluate $I_c(R\rangle B_1\ldots B_n)$ on



$I_R \otimes N^{\otimes n}(|\psi\rangle\langle\psi|)$.

$$\boxed{\begin{array}{l}\text{The LSD thm:} \\ Q(N) = \sup_n Q^{(n)}(N).\end{array}}$$

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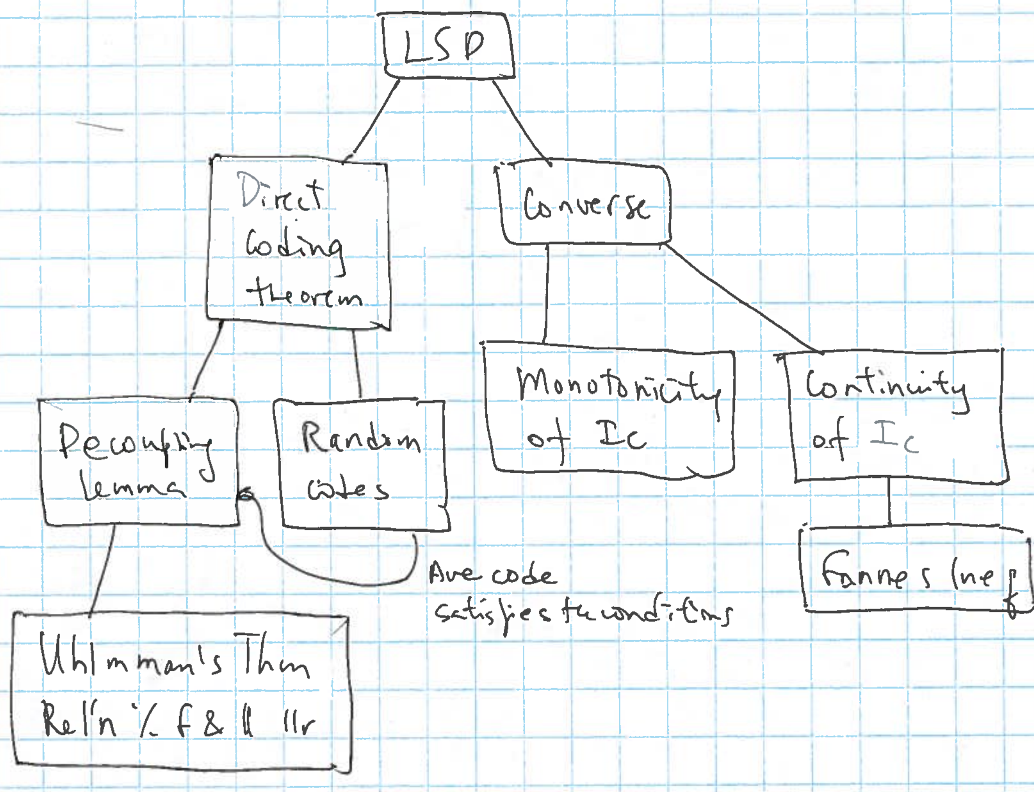
NB:

- When transmitting classical data, we maximize $S(B:R)$ for $\rho_{RB} = \sum_x p_x |x\rangle\langle x|_R \otimes N^{\otimes n}(\rho_x)_B$
- When transmitting quantum data, we maximize $S(B:R) - S(B:E)$ for $|\psi\rangle_{RBE} = I \otimes U^{\otimes n} |\psi\rangle_{RA}$
- Bob has to "spend" his correlation with R to uncorrelate E's with R – a necessary condition for near perfect quantum communication.
- Proves will use def (2a), that the channel can establish Max ent states

<u>Proving the LSD thm</u> (ref 07.2005 or Preskill notes May18-Jun1, 2009)



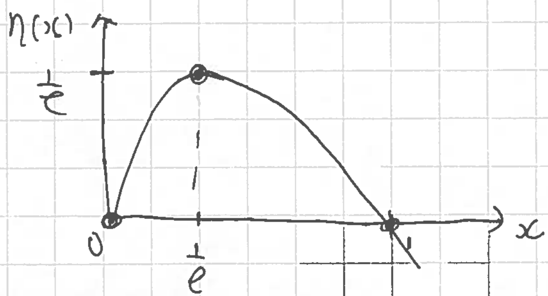
Fannes Inequality

Let $\rho_1, \rho_2 \in D(\mathbb{C}^d)$ [density matrices on d-dim HS].

$$\|\rho_1 - \rho_2\|_1 = \Delta \le \frac{1}{e}$$

Then $|S(\rho_1) - S(\rho_2)| \le \Delta \log d - \eta(\Delta)$

where $\eta(x) = -x \log x$.



① Both terms $\to 0$ as $\Delta \to 0$.

② 2nd term dim-independent.

Pf: Watrous CS766 F2011 Lec10

or NC P512 [Note there, $T(\rho,\sigma) = \|\rho-\sigma\|_1 = 2\|\rho-\sigma\|_{tr} = 2D(\rho,\sigma)$].

Corollary:

Let $\rho_1, \rho_2 \in D(\overset{R}{\mathbb{C}^d} \otimes \overset{B}{\mathbb{C}^d})$

$$\| \rho_1 - \rho_2 \|_1 = \Delta \le \frac{1}{e}$$

Then $\left| I_c(R \rangle B)_{\rho_1} - I_c(R \rangle B)_{\rho_2} \right| \le 3 \Delta \log d + 2\eta(\Delta)$

Pf: $\left| I_c(R \rangle B)_{\rho_1} - I_c(R \rangle B)_{\rho_2} \right|$

$$= \left| [S(B) - S(RB)]_{\rho_1} - [S(B) - S(RB)]_{\rho_2} \right|$$

$$\le | S(B)_{\rho_1} - S(B)_{\rho_2} | + | S(RB)_{\rho_1} - S(RB)_{\rho_2} |$$

$$\le \Delta \log d + \Delta \log(d^2) + 2\eta(\Delta)$$

$\because \| \rho_{1B} - \rho_{2B} \|_1 \le \| \rho_1 - \rho_2 \|$

$$= 3 \Delta \log d + 2\eta(\Delta).$$

NB: Most entropy quantities are continuous for similar reasons, except for $S(\rho \| \sigma)$, I_{acc}, or when dim term overwhelm the proximity bd.

Converse:

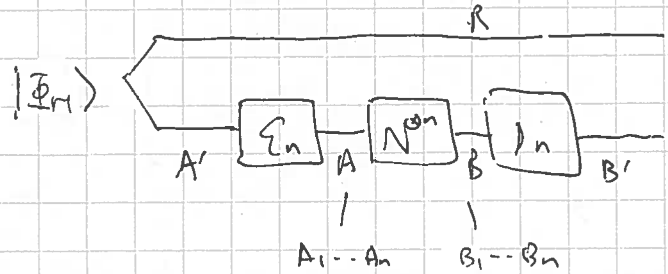
Suppose r achievable (lower case r).

$\forall \varepsilon > 0$, $\exists n$ and an (M, n) code, $M = \lfloor 2^{nr} \rfloor$,

s.t. $I \otimes D_n \circ N^{\otimes n} \circ \mathcal{E}_n(\Phi_M) \underset{\varepsilon}{\approx} \Phi_M$ ← used def (2a) or (2b).

take trace distance

$$|\Phi_M\rangle = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} |i\rangle|i\rangle$$



We have:

$$I_c(R \rangle B)_{I \otimes N^{\otimes n}(\mathcal{E}_n(\Phi_M))}$$

$$\overset{\text{mono } I_c}{\geq} I_c(R \rangle B')_{I \otimes D_n \otimes N^{\otimes n}(\mathcal{E}_n(\Phi_M))}$$

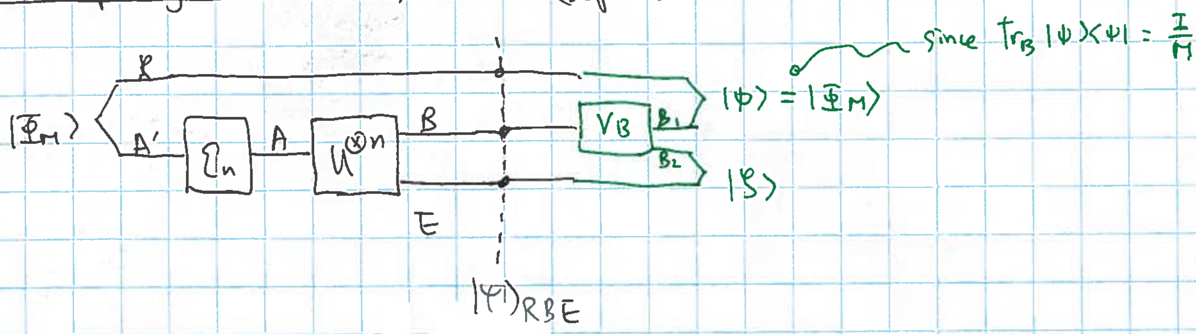
$$\overset{\text{cty } I_c}{\geq} I_c(R \rangle B')_{\Phi_M} + 3 \cdot (2\varepsilon) \cdot \log M + 2\eta(2\varepsilon)$$

$$= nr + 6\varepsilon n r + 2\eta(2\varepsilon).$$

$$\max_{|\psi\rangle} \frac{1}{n} I_c(R \rangle B)_{I \otimes N^{\otimes}(|\psi\rangle\langle\psi|)} \geq \frac{1}{n} I_c(R \rangle B)_{I \otimes N^{\otimes}(\mathcal{E}_n(\Phi_M))} \geq r + 6\varepsilon r + \frac{1}{n} 2\eta(2\varepsilon)$$

$$\therefore Q^{(n)}(N) - 6\varepsilon r - \frac{1}{n} 2\eta(2\varepsilon) \geq r.$$

Decoupling (exact case): (before $\mathcal{D}_n$)



NB: E purifies RB & $\underbrace{\text{B purifies RE.}}_{|\psi\rangle_{RBE} \text{ purifies } \rho_{RE}}$

Observation:

- If $\rho_{RE} = \rho_R \otimes \rho_E$

  then we have another purification: $|\phi\rangle_{RB_1} \otimes |\xi\rangle_{EB_2}$

  where $B = B_1 B_2$, $|\phi\rangle_{RB_1}$ purifies ρ_R
  $|\xi\rangle_{EB_2}$ ... ρ_E.

- Recall any 2 purifications differ by a unitary on the purifying system. (Sec 2.5 NC Ex 2.81).

- $\exists V_B$ s.t. $(V_B \otimes I_{RE})|\psi\rangle_{RBE} = |\phi\rangle_{RB_1} \otimes |\xi\rangle_{EB_2}$

- See diagram

Interpretation: purification of R is BE, and since it's "outside of E", it's completely in B.

Decoupling (exact) $\Rightarrow$ $\mathcal{E}_n$ works & $\frac{1}{n}\log M$ achievable.

Approx decoupling $\Rightarrow$ approx transmission of $|\Phi_M\rangle$

<u>Lemma</u>:

If $\left\| \rho_{RE} - \left(\frac{I}{M}\right)_R \otimes \rho_E \right\|_1 \leq \varepsilon'$

then $\exists\ |\psi\rangle_{REB}$ purifying ρ_{RE}, for $B = B_1 B_2$

s.t. $\left\| tr_{EB_2} |\psi\rangle\langle\psi|_{REB} - |\Phi_M\rangle\langle\Phi_M|_{RB_1} \right\|_1 \leq 2\sqrt{\varepsilon'}$

<u>Pf</u>:

- Recall $1 - F(\mu, \nu) \leq \frac{1}{2} \| \mu - \nu \|_1$

$\therefore\ F\left(\rho_{RE}, \left(\frac{I}{M}\right)_R \otimes \rho_E\right) \geq 1 - \frac{\varepsilon'}{2}$

since any $|\beta\rangle = U \otimes I |\beta_0\rangle$, absorb U into $|\alpha\rangle$

- Recall [Uhlmann] $F(\mu, \nu) = \max\limits_{\substack{|\alpha\rangle, |\beta\rangle \\ \text{purifies } \mu \quad \text{purifies } \nu}} |\langle\alpha|\beta\rangle| = \max\limits_{\substack{|\alpha\rangle \\ \text{purifies } \mu}} |\langle\alpha|\beta_0\rangle|$

fixed purification of σ.

- Choose $|\beta_0\rangle = |\Phi\rangle_{RB_1} \otimes |\varsigma\rangle_{B_2E}$

$\exists\ |\alpha\rangle = |\psi\rangle_{REB}$

s.t. $1 - \frac{\varepsilon'}{2} \leq F\left(\rho_{RE}, \left(\frac{I}{M}\right)_R \otimes \rho_E\right) = F\left(|\Phi_M\rangle_{RB_1} \otimes |\varsigma\rangle_{B_2E}, |\psi\rangle_{REB}\right)$

- Recall $\frac{1}{2} \|\mu - \nu\|_1 = \sqrt{1 - F(\mu, \nu)^2}$ if μ, ν pure.

$\therefore\ \left\| |\psi\rangle\langle\psi|_{REB} - |\Phi_M\rangle\langle\Phi_M|_{RB_1} \otimes |\varsigma\rangle\langle\varsigma|_{B_2E} \right\|_1$

$= 2\sqrt{1 - F(|\psi\rangle_{REB}, |\Phi_M\rangle_{RB_1} |\varsigma\rangle_{B_2E})^2}$ $\quad [(1-f^2) = (1-f)(1+f)]$

$\leq 2\sqrt{2}\sqrt{1-F} \leq 2\sqrt{\varepsilon'}$.

Tracing out EB_2, $\| \ \|_1$ non increasing.