Lec 13

Remarks on approximating channels.

For 2 general channels N_1 & N_2 :

① $\forall |\psi\rangle$, $\| N_1(|\psi\rangle\langle\psi|) - N_2(|\psi\rangle\langle\psi|) \|_{tr} \leq \varepsilon$

$\not\Rightarrow \| N_1 - N_2 \|_\diamond$ small

①' [$\Rightarrow$] holds if N_2 = Identity channel.

② $\| I \otimes N_1 (|\Phi\rangle\langle\Phi|) - I \otimes N_2 (|\Phi\rangle\langle\Phi|) \|_{tr} \leq \varepsilon$

$\not\Rightarrow \| N_1 - N_2 \|_\diamond$ small

(1') was shown in 9809010 Barnum, Knill, Nielsen:

If $\langle\psi| N(|\psi\rangle\langle\psi|)|\psi\rangle \geq 1-\eta \quad \forall|\psi\rangle$

Then $\langle\psi| I\otimes N(|\psi\rangle\langle\psi|)|\psi\rangle \geq 1-\frac{3\eta}{2} \quad \forall|\psi\rangle$

Using rel'n between fidelity & trace distance,

the above implies (1').

(1) was shown in 0307104 Hayden, L, Shor, Winter:

If N_1 = complete randomization channel, ie $N_1(\rho) = \frac{I}{d} \quad \forall\rho$

d = output dim

$N_2(\rho) = \frac{1}{n}\sum_{k=1}^{n} U_k \rho U_k^\dagger$ where $n = \frac{134\, d \log d}{\varepsilon^2}$

U_k chosen ~ Haar measure.

then whp, " $\forall|\psi\rangle, \| N_2(|\psi\rangle\langle\psi|) - \frac{I}{d} \|_{tr} \leq \varepsilon$ " (such N_2 exists)

But $\| \underbrace{I\otimes N_1(\Phi)}_{\frac{I}{d}\otimes\frac{I}{d}} - \underbrace{I\otimes N_2(\Phi)}_{\text{mixture of } O(d\log d) \text{ states}} \|_{tr} \approx 1$

$\therefore \| N_1 - N_2 \|_\diamond \geq \| \quad \| \approx 1$.

To see (2):

Let $U = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix}$. $\quad |\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$

So $I \otimes U \, |\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d-1} |ii\rangle - \frac{1}{\sqrt{d}} |dd\rangle = |\Phi'\rangle$

Let $N_2(\rho) = U \rho U^\dagger$.

$N_1(\rho) = \rho$.

Then, $\| \overbrace{I \otimes N_1(\Phi) - I \otimes N_2(\Phi)}^{\text{both pure}} \|_{tr}$

$= \sqrt{1 - |\langle \Phi | \Phi' \rangle|^2}$

$= \sqrt{1 - \left(\frac{d-2}{d}\right)^2} \leq \frac{2}{\sqrt{d}}$

Consider the state $|\psi\rangle_{RA} = |0\rangle_R \left(\frac{|1\rangle + |d\rangle}{\sqrt{2}} \right)_A$.

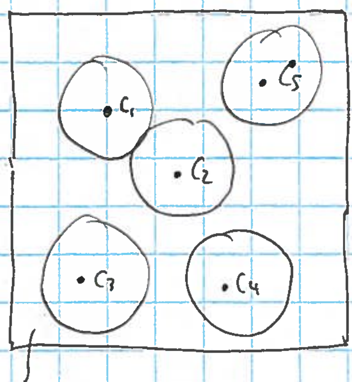
$$\| N_1 - N_2 \|_\diamond \geq \| \underbrace{I \otimes N_1(|\psi\rangle\langle\psi|)}_{|0\rangle_R \left(\frac{|1\rangle + |d\rangle}{\sqrt{2}}\right)_A} - \underbrace{I \otimes N_2(|\psi\rangle\langle\psi|)}_{|0\rangle_R \left(\frac{|1\rangle - |d\rangle}{\sqrt{2}}\right)_A} \|_{tr} = 1.$$

So (2) fails even if $N_1 = $ Id map.

☆ Luckily all 4 def of errors / approx of identity give same capacity. We only demand a near noiseless subspace.

What's the quantum (M, n) code?

Classical ECC $(M=5)$



Ambient space

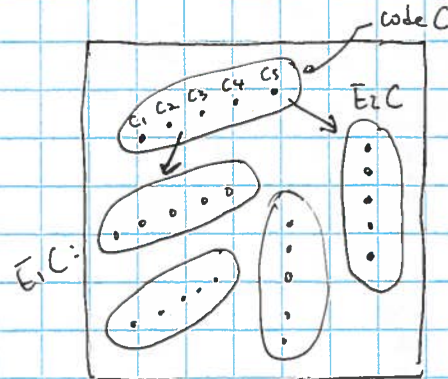
Codewords: subset

Condition for ECC:

Barely overlapping error spheres.

"measure the data"

Alternative view:



Condition for ECC:

Errors to be corrected (the common ones) $E_1, E_2, \ldots$ Take the code C to barely overlapping images, so one can find the "error syndrome"

Also, E_i's should be invertible.

Note: distortion in distance.

"measure the error"

Quantum ECC

the same

Except it is possible $E_i C = E_j C$ for $i \neq j$

"Degenerate code"

Ex check on the 3 bit rep code, for up to 2 errors.

eg. Let $N(\rho) = (1-p)\rho + p\frac{I}{2}$

$$= (1-p)\rho + p \cdot \tfrac{1}{4}(\rho + X\rho X + Z\rho Z + Y\rho Y)$$

Pauli's

$$= \left(1 - \tfrac{3p}{4}\right)\rho + \tfrac{p}{4}(X\rho X + Z\rho Z + Y\rho Y).$$

Goal: correct for up to 1 Pauli error

Code #1: 9 bit degenerate code

$$|0\rangle_L = \frac{1}{2^{3/2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|1\rangle_L = \frac{1}{2^{3/2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

$a|0\rangle + b|1\rangle \downarrow a|0\rangle_L + b|1\rangle_L$

- Compare if qubits 12, 23, 45, 56, 78, 89 all have even parity
- Compare if blocks 12, 23 have same sign.

(Give example after X_4, Z_9.)

- Note Z_1, Z_2, Z_3 act indistinguishably

Similarly for Z_4, Z_5, Z_6.
Z_7, Z_8, Z_9.

No need to distinguish which of Z_1, Z_2, Z_3 occurs since they are all reverted by Z_1

Stabilizer formalism:

Code C is a subspace of $(\mathbb{C}^2)^{\otimes 9}$.

C is the "+1" eigenspace of <u>commuting</u> operators: (observables):

$G_1 = XXX \quad XXX \quad 111$ ← +1 eigenspace has matching signs in blocks 1, 2.
$G_2 = 111 \quad XXX \quad XXX$ ← 2, 3

$G_3 = ZZ1 \quad 111 \quad 111$ ← +1 - - - - - even parity on qubits 12
$\quad\ \ 1ZZ \quad 111 \quad 111$ ← 23
$\quad\ \ 111 \quad ZZ1 \quad 111$
$\quad\ \ 111 \quad 1ZZ \quad 111$
$\quad\ \ 111 \quad 111 \quad ZZ1$
$G_8 = 111 \quad 111 \quad 1ZZ$

+1 eigenspace of each "generator" has $\frac{2^9}{2}$ dims

but simultaneous +1 eigenspace -- $\frac{2^9}{2^8} = 2$ dim.

- Say $\underbrace{X_1(C)}_{\text{space after error } X_1}$ is in -1 eigenspace of G_3
  $+1$ - - of the rest.

Other errors have different "patterns of ± 1" (error syndrome).

- $Z_4(C)$, $Z_5(C)$, $Z_6(C)$ all in -1 eigenspace of G_1, G_2
  $+1$ - - rest

NB: Can learn syndrome by measurement $\{\Pi^+_{G_i}, \Pi^-_{G_i}\}$ $\forall i$

(or a 2^8-outcome projective measure each with rank 2.)

the 8-bit eigenvalue information

Code #2   5 bit nondegenerate code

Specified by 4 generators

$G_1 = X\ Z\ Z\ X\ 1$

$G_2 = 1\ X\ Z\ Z\ X$

$G_3 = X\ 1\ X\ Z\ Z$

$G_4 = Z\ X\ 1\ X\ Z$

Note commuting.

Encodes 1 qubit $\left(\frac{2^5}{2^4} = 2 \text{ dims}\right)$.

Errors to be corrected:

$I, X_1, Y_1, Z_1, X_2, \ldots, X_5, Y_5, Z_5$

16 cases.

Each case corresponds to a unique error syndrome (the 4 bit eigenvalue info).

e.g. If we sees $+ + - +$ for $G_{1\,2\,3\,4}$

error is Z_3.

This code saturates the Hamming bound

$\because \underset{\#\text{ errors}}{16} \times \underset{\text{code dim}}{2} = 32 =$ dim of ambient space.

if error E anticommutes with G_i

then $G_i(E|\psi\rangle)$ for $|\psi\rangle \in C$

$= -E(G_i|\psi\rangle)$

$= -E|\psi\rangle$

$\therefore$ E takes C to a -1 ~~subspace with 1~~ eigen subspace of G_i