~~Last time~~

Lec 12.

HSW Thm:

$$C(N) = \sup_n \chi^{(n)}(N)$$

where $\chi^{(1)}(N) = \max_{\{p_x, \rho_x\}} \chi(\{p_x, N(\rho_x)\})$

$$\chi^{(n)}(N) = \frac{1}{n}\chi^{(1)}(N^{\otimes n}).$$

Back to $C(N)$.

$C(N)$ is only known for N s.t. $\chi^{(1)}(N) = \chi^{(n)}(N)$.

Def: χ strongly additive on N if

$$\forall N', \quad \chi^{(1)}(N \otimes N') = \chi^{(1)}(N) + \chi^{(1)}(N') \quad — \star .$$

Def: χ weakly additive on N if $\chi^{(n)}(N) = \chi^{(1)}(N)$.

Obs: χ strongly additive on $N \Rightarrow$ weakly additive on N.

Obs: $\geq$ in $(\star)$ is obvious.

$=$ if restricted to optimize over ensembles of product states on LHS of $\star$

Known additivity results:

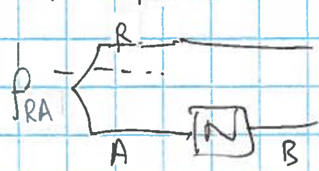
① If N entanglement breaking, then χ strongly additive on N (0302031).

Cor: Then $C(N) = \chi^{(1)}(N)$.

Def: N is entanglement breaking

if $\forall \rho_{RA}$, $I \otimes N(\rho_{RA})$ is not entangled (separable).

a convex combination of product states.



← not entangled with R.

Characterization: N ent breaking $\Leftrightarrow$ $N =$ "QCQ" channel.

$$\text{ie } \rho \to \boxed{N} \to \; = \; \rho \to \boxed{\boxed{M} \xrightarrow{y} \boxed{Q}} \to \rho_y$$

eg classical channels are ent breaking

Intuition: Entangled ensemble required for superadditivity.

(2) If N is unital (ie $N(I) = I$) and N has 2 dim input & output, then χ is strongly additive on N.

(Due to King 0103156).

Cor: $C(N) = \chi^{(1)}(N)$ for qubit unital channels

- An interesting side fact:

All qubit unital channels have the form:

$$\rightarrow \boxed{U} - \boxed{\mathcal{E}} - \boxed{V} \rightarrow$$

Pauli's.

where $U, V \in \mathcal{U}(2)$, and $\underbrace{\mathcal{E}(\rho) = \sum_i p_i \sigma_i \rho \sigma_i}_{\text{random Pauli channel.}}$

(3) If N is the d-dim depolarizing channel, then, χ is strongly additive on N.

- The channel in eg 1 & the channel in the exercise after both have $\chi^{(1)}(N) = C(N)$. Thus our calculations for $\chi^{(1)}(N)$ give $C(N)$ as well.

<u>Unknown additivity:</u>

χ is not known to be additive on the AD channel N_γ

So we don't know $C(N_\gamma)$.

Non additivity :

Hastings 0809.3972 shows that

$\exists N_1, N_2$ s.t. $\chi^{(1)}(N_1 \otimes N_2) \underset{\neq}{\geq} \chi^{(1)}(N_1) + \chi^{(1)}(N_2)$.

Before this, there were 4 "additivity conjectures" that turn out "equivalent."

Def : Entanglement of formation

$$E_F(\rho) = \min_{\underbrace{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|}_{\text{to be optimized}}} \sum_i p_i \underbrace{E(|\psi_i\rangle)}_{\substack{\text{pure state entanglmt} \\ = S(\mathrm{tr}_A |\psi_i\rangle\langle\psi_i|)}}$$

ρ given

average

Min Average entanglement in a pure state decomposition of ρ.

Intuitive method to make $\rho^{\otimes n}$:

(1) Use ebits to create $\approx n p_i$ copies of $|\psi_i\rangle$ $\forall i$.

- entanglement "dilution" requires $\approx n E(|\psi_i\rangle)$ ebits & $O(\sqrt{n})$ cbits to create n copies of $|\psi_i\rangle$.

(2) Alice draws i wp p_i, iid n times.

Gets $i_1 i_2 \ldots i_n$, shares outcome with Bob.

(3) They make a copy of $|\psi_{i_1}\rangle$ in $A_1 B_1$,
$|\psi_{i_2}\rangle$ in $A_2 B_2$
$\vdots$
$|\psi_{i_n}\rangle$ in $A_n B_n$,

forget $i_1 \ldots i_n$, and share $\rho^{\otimes n}$.

[Thus "formation".]

$\approx$ 1996 (BDSW?)

Qn: Is $E_f(\rho)$ the min asymptotic # ebits needed for copy of ρ?

In 0008134, Hayden, Horodecki, and Terhal defined the entanglement cost:

$$E_C(\rho) = \lim_{\substack{n\to\infty \\ \epsilon_n \to 0}} \frac{1}{n}\left(\begin{array}{l}\text{\# ebits needed to product } \rho^{\otimes n} \\ \text{with accuracy } \epsilon_n \text{ \& w/ free cbits}\end{array}\right)$$

and proved:

$$E_C(\rho) = \lim_{n\to\infty} \frac{1}{n} E_f(\rho^{\otimes n}) \leq E_f(\rho)$$

$\neq$ if $\rho^{\otimes n} = \sum_j q_j \underbrace{|\phi_j\rangle\langle\phi_j|}_{\text{entangled over } A_1B_1A_2B_2 \cdots A_nB_n}$

gives decomposition better than

$\rho^{\otimes n} = \left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right)^{\otimes n}$

one copy in each of $(A_1B_1)(A_2B_2)\cdots(A_nB_n)$

all true or all false

In 0305035, Shor prove TFAE:

① $\forall N_1, N_2$ $\quad \chi^{(1)}(N_1 \otimes N_2) = \chi^{(1)}(N_1) + \chi^{(2)}(N_2)$

② $\forall \rho_1, \rho_2$, $\quad E_F(\rho_1 \otimes \rho_2) = E_F(\rho_1) + E_F(\rho_2)$

③ $\forall N_1, N_2$, $\quad S_{min}(N_1 \otimes N_2) = S_{min}(N_1) + S_{min}(N_2)$ $\quad$ ← what Hastings disproved. (Lec 18 S 2010)

[where $S_{min}(N) = \min_\rho S(N(\rho))$ = min output entropy]

④ $\forall \rho_{A_1B_1A_2B_2}$, $\quad E_F(\rho_{A_1A_2|B_1B_2}) \geq E_F(\rho_{A_1B_1}) + E_F(\rho_{A_2B_2})$

(Lecs 8-9 S 2010)

How was ③ disproved?

References: Hastings 0809.3972

- Also Brandao, (M) Horodecki 0907.3210
  Fukuda, King, Moser 0905.3697
  Aubrun, Szarek, (E) Werner 0910.1189.

- Built on failure of ③ for Renyi-entropies
  Holevo-Werner (R), Winter, Hayden

<u>Main ideas:</u>

- Let input dim = output dim = d.
- Choose N_1: $N_1(\rho) = \frac{1}{n} \sum_{k=1}^{n} U_k \rho U_k^{\dagger}$, $U_k \in \mathcal{U}(d)$

  $N_2 = \overline{N_1}$: $N_2(\rho) = \frac{1}{n} \sum_{k=1}^{n} \overline{U_k} \rho \overline{U_k}^{\dagger}$. "$-$" complex conjugate.

- Show $S_{min}(N_1 \otimes N_2) \lesssim 2 \log n - \frac{\log n}{n}$

  Reason:
  Joint input $|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle|i\rangle$ well preserved.

  Reason:

  Transpose trick: $A \otimes I |\Phi\rangle = I \otimes A^T |\Phi\rangle$ $\forall$ $d \times d$ matrices.
  Pf: Ex (1 line).

  Thus $U \otimes \bar{U} |\Phi\rangle = I \otimes U^T \bar{U} |\Phi\rangle = |\Phi\rangle$.

$\therefore$ n out of n^2 Kraus ops in $N_1 \otimes N_2$ fix $|\Phi\rangle$

$\therefore$ $\langle \Phi | N_1 \otimes N_2 (|\Phi\rangle\langle\Phi|) |\Phi\rangle \geq \frac{1}{n}$.

Worse case spectrum : $\{ \frac{1}{n}, \underbrace{(1-\frac{1}{n}) \frac{1}{n^2-1}, \ldots (1-\frac{1}{n}) \frac{1}{n^2-1}}_{n^2-1 \text{ times}} \}$.

(Note $N_1 \otimes N_2 (|\Phi\rangle\langle\Phi|)$ rank $\leq n^2$.)

$$\Rightarrow S_{min} (N_1 \otimes N_2) \leq S(N_1 \otimes N_2 (|\Phi\rangle\langle\Phi|))$$

$$\leq -\frac{1}{n} \log \frac{1}{n} - (1-\frac{1}{n}) \log (1-\frac{1}{n^2}) \left(\frac{1}{n^2-1}\right)$$

$$\leq 2 \log n - \frac{\log n}{n}.$$

- Note $S_{min} (N_1) = S_{min} (N_2)$.

  Since $S(N_1(\rho)) = S(N_2(\bar{\rho}))$

  Difficult part:

  $$\forall |\psi\rangle, \; S(N_1 (|\psi\rangle\langle\psi|)) \geq \log n - \left[\frac{\text{const}}{n} + \text{poly}(n) \, O \sqrt{\frac{\log d}{d}} \right]$$

  Requires choosing U_k's according to Haar measure.

  Assume $\exists$ input with low entropy output & get contradiction

- Choose $d \ggg n$ gives $S_{min} (N_1 \otimes N_2) \underset{\neq}{<} 2 \, S_{min} (N_1)$.

Note large dim, nonconstructive, variations on N_1, & proof method.
tight race $\frac{1}{\cdot}$ 2 terms.

Transmitting quantum data via noisy quantum channel:

Concepts concerning QECC's & noise suppression & asymptotics similar to before.

Def: Given quantum channel $\mathcal{N}$, we say R is an achievable rate if $\forall n$, $\exists\, (M, n)$ code transmitting M-dim quantum states with n uses of $\mathcal{N}$, with $M = 2^{nR}$ & error $\varepsilon_n \to 0$.

$$\rho \xrightarrow[M=2^{nR}\text{ dims}]{} \underbrace{\boxed{\mathcal{E}_n}}_{\text{encoder}} \to \boxed{\mathcal{N}^{\otimes n}} \to \underbrace{\boxed{\mathcal{D}_n}}_{\text{decoder}} \to \tilde{\rho} \overset{\varepsilon_n}{\approx} \rho .$$

Def: Quantum capacity $Q(\mathcal{N})$ = supremum over achievable rates.

Qn: How to define $\tilde{\rho} \overset{\varepsilon_n}{\approx} \rho$?

Recall $\| \ \|_{tr}$ & fidelity.

Consider the following measures of proximity:

① $|\psi\rangle \rightarrow \boxed{\mathcal{E}_n} - \boxed{N^{\otimes n}} - \boxed{\mathcal{D}_n} - \hat{\rho}$.

ⓐ Ave $\langle\psi|\hat{\rho}|\psi\rangle \geq 1-\varepsilon_n$ — Ave fidelity

ⓑ min $\langle\psi|\hat{\rho}|\psi\rangle \geq 1-\varepsilon_n$ — Worse case fidelity

②

ⓐ $\langle\psi|\tilde{\rho}|\psi\rangle \geq 1-\varepsilon_n$ for $|\psi\rangle$ = max ent state

ⓑ $\langle\psi|\tilde{\rho}|\psi\rangle \geq 1-\varepsilon_n$ $\forall|\psi\rangle$. — Worse case Entanglement fidelity

Or use trace distance instead of fidelity in any of the above.

All 4 defs

* Give same $Q(N)$!

See 9809010, 0311037 for more

Barnum, Knill, Nielsen — Kretschmann, (R) Werner

Def (1) revolves about preserving the "input"
Easiest to think about & to show $\varepsilon_n \to 0$

Def (2) revolves about $D_n \circ N^{\otimes n} \circ \mathcal{E}_n$ simulating the identity channel on 2^{nR}-dim Hilbert space.

When def (2) is used with the trace distance criteria:

(2a) bounds the trace distance between the "Choi-Jamiołkowski" states of the simulating & simulated channels.

CJ state of $N = (I \otimes N)(|\Phi\rangle\langle\Phi|)$ where $|\Phi\rangle = \sum_i |i\rangle|i\rangle$.

(2b) bounds the distance between the simulated & simulating channels induced by the diamond norm.

$$\| N_1 - N_2 \|_\diamond := \max_{|\psi\rangle} \| I \otimes N_1 (|\psi\rangle\langle\psi|) - I \otimes N_2 (|\psi\rangle\langle\psi|) \|_{Tr}$$

The most stringent meas that N_1 is similar to N_2.
No one can tell whether N_1 or N_2 has been applied with bias $\frac{1}{2} + \frac{1}{2} \| N_1 - N_2 \|_\diamond$.

Def (2b) useful if we already know a channel capacity.

In general, given 2 channels N_1, N_2

① $\forall |\psi\rangle$, $\| N_1(|\psi\rangle\langle\psi|) - N_2(|\psi\rangle\langle\psi|) \|_{tr} \leq \varepsilon$ — (a)

$\not\Rightarrow$ $\|N_1 - N_2\|_\diamond$ small

② $\| I \otimes N_1(|\Phi\rangle\langle\Phi|) - I \otimes N_2(|\Phi\rangle\langle\Phi|) \|_{tr} \leq \varepsilon$ — MES.

$\not\Rightarrow$ $\|N_1 - N_2\|_\diamond$ small.

Let input & output dims of both N_1, N_2 be d. Problem worse with large d (the case we care here).

① was first demonstrated in 0307104 :

Take N_1 = completely randomizing channel.
i.e. $N_1(\rho) = \frac{I}{d}$ $\forall \rho$. NB N_1 has d^2 Kraus ops, $I \otimes N_1(|\Phi\rangle\langle\Phi|) = \frac{I}{d} \otimes \frac{I}{d}$.

Take $N_2(\rho) = \frac{1}{n} \sum_{k=1}^{n} U_k \rho U_k^\dagger$ where $n \approx d \log d$

With high prob, condition (a) holds, but N_2 very diff from N_1.

✩ NB : "$\Rightarrow$" holds if N_1 = identity channel. (9809010)

② Take $N_1(\rho) = \Pi \rho \Pi + \frac{|d\rangle\langle d|}{d-1} \langle d|\rho|d\rangle$.

Take $N_2(\rho) = \Pi \rho \Pi + \frac{|1\rangle\langle 1|}{d-1} \langle d|\rho|d\rangle$, $\left(\Pi = \sum_{k=1}^{d-1} |k\rangle\langle k| \right)$

Then $\| I \otimes N_1(|\Phi\rangle\langle\Phi|) - I \otimes N_2(|\Phi\rangle\langle\Phi|) \|_{tr}$

$= \left\| \frac{1}{d-1} \left(|d\rangle\langle d| \otimes |d\rangle\langle d| - |d\rangle\langle d| \otimes |1\rangle\langle 1| \right) \right\|_{tr}$

$= \frac{2}{d-1}$

But $\| N_1 - N_2 \|_\diamond = 1$. Choose input $|d\rangle\langle d| \otimes |d\rangle\langle d|$.