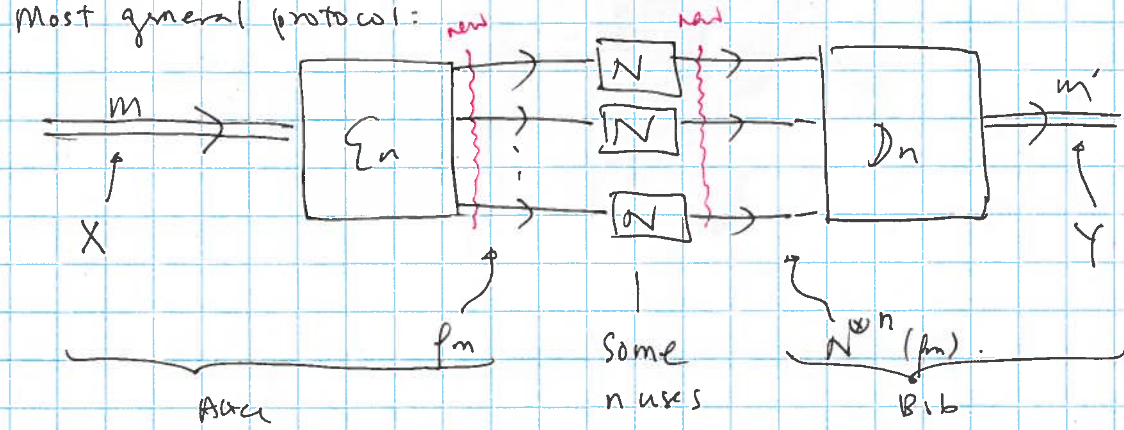
Classical capacity of a quantum channel:

Most general protocol:



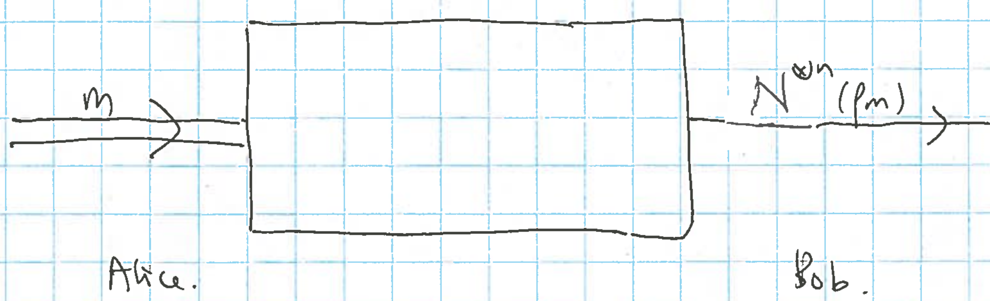
Alice & Bob agree on n, a ρ_m for each m [ie $\mathcal{E}_n$ is known.]

(1) If rate R achievable:

$$nR \lesssim I(X:Y) \le I_{acc}\left(\{p_m, \mathcal{N}^{\otimes n}(\rho_m)\}\right).$$

$$\le \max_{p_m, \rho_m} \chi\left(\{p_m, \mathcal{N}^{\otimes n}(\rho_m)\}\right).$$

(2) But for any choice of ρ_m's:



is a Q-box, and using Q-box l times (l large), can achieve rate (per box)

$$nR = R_Q \ge \max_{p_m} \chi\left(\{p_m, \mathcal{N}^{\otimes n}(\rho_m)\}\right)$$

Thm [Holevo-Schumacher-Westmoreland]: (HSW Thm)

$$C(N) = \sup_n \frac{1}{n} \chi(N^{\otimes n}).$$

where $\chi(\eta) := \max_{\{p_x, \rho_x\}} \chi(\{p_x, \eta(\rho_x)\})$

Holevo: IEEE TIT 44, 269 (1998)
Schumacher & Westmoreland: PRA 56, 131 (1997).

NB: n not # uses in above.

n, p_x, ρ_x → what we optimize over

NB: The optimization over n makes the expression "regularized"

This is in contrast to the capacity expression for the channel channel or the Q-box; the optimizations involve only 1 copy of channel.

When proving converse for $C(Q)$

$$S(B_1 B_2 \ldots B_n \mid X_1 X_2 \ldots X_n)$$

$$= \sum_{x_1 x_2 \ldots x_n} p(x_1 \ldots x_n)\, S(\rho_{x_1} \otimes \rho_{x_2} \otimes \ldots \otimes \rho_{x_n}).$$

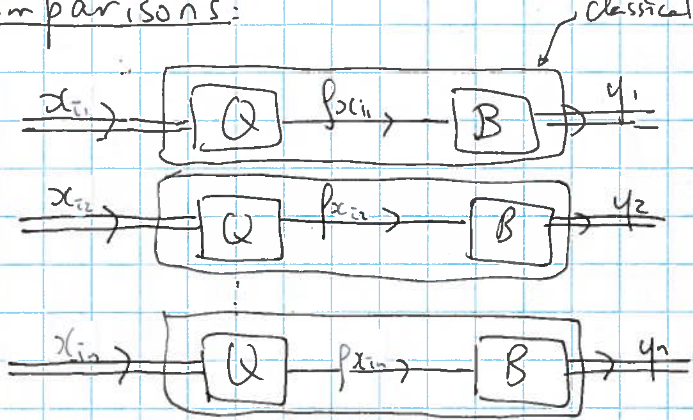
breaks in HSW

$$= \sum_{x_1 x_2 \ldots x_n} p(x_1 \ldots x_n) \sum_i S(\rho_{x_i})$$

message = i

Comparisons:

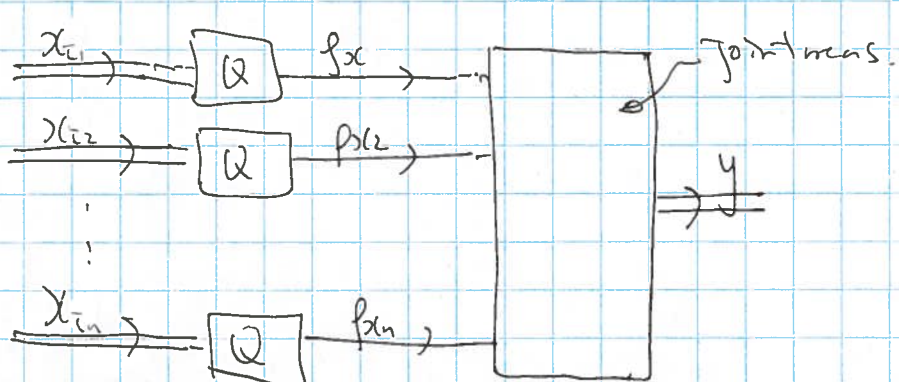
①



Capacity = $\max\limits_{p_x} \; I_{acc}(\{p_x, \rho_x\})$.

Shannon's Thm, code $= \{c_i = x_{i1} x_{i2} \ldots x_{in} \;;\; x_{ij} \sim p(x)\}$.

②

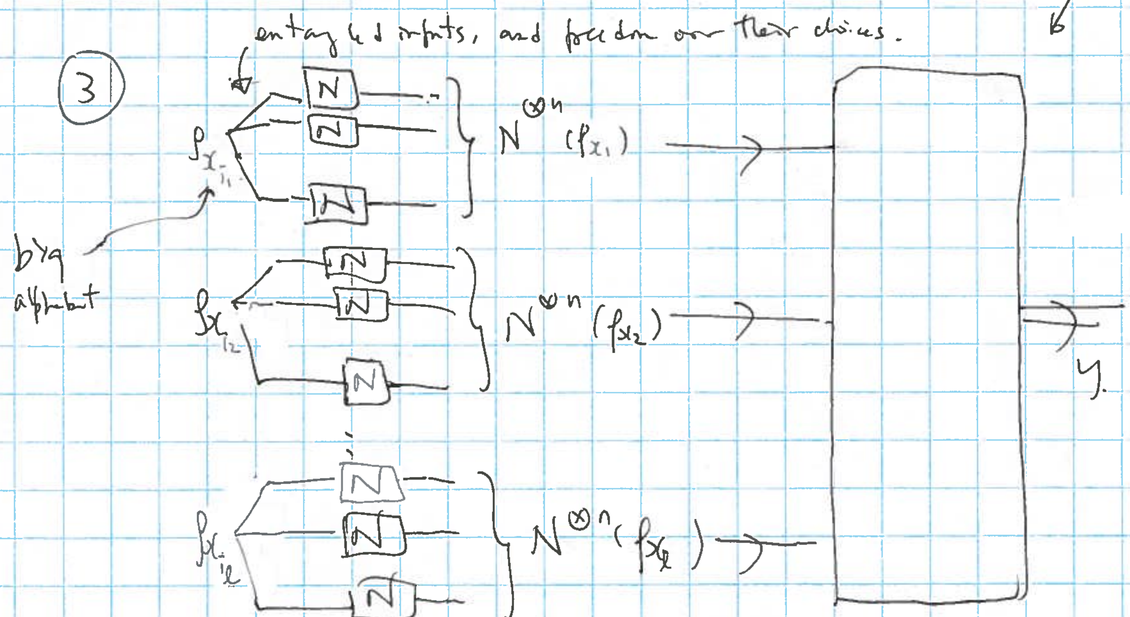


Capacity = $C(Q) = \max\limits_{p_x} \; \chi(\{p_x, \rho_x\})$,

Code = as above, restrict to strongly typical sequences.

Note double blocking

③



max over $\{p_x, \rho_x\}$

Capacity $= C(N)$

$= \sup\limits_n \frac{1}{n} \chi(p_x, N^{\otimes n}(\rho_x))$

Code $= \rho_{x_{i1}} \otimes \rho_{x_{i2}} \otimes \cdots \otimes \rho_{x_{il}}$

$\underbrace{c_i = x_{i1} x_{i2} \cdots x_{il}}_{\text{random but strongly typical.}}$

Consequences of the HSW thm:

Define $\chi(N) = \chi^{(1)}(N) = \max_{\{p_x, \rho_x\}} \chi(\{p_x, N(\rho_x)\})$

[choose input ensemble, calculate χ on output ensemble]

$$\chi^{(n)}(N) = \frac{1}{n} \chi(N^{\otimes n})$$

① $C(N) = \sup_n \chi^{(n)}(N) \geq \chi^{(n)}(N) \geq \underbrace{\chi^{(1)}(N)}_{\text{useful lower bound.}}$

② $C(N) = 0$ iff $\chi^{(1)}(N) = 0$

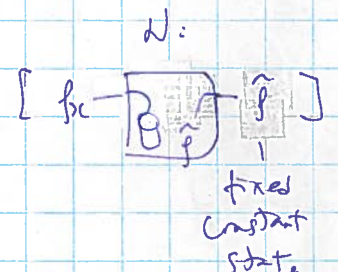
[$\Rightarrow$] obvious

[$\Leftarrow$] $\chi^{(1)}(N) = \max_{p_x, \rho_x} S(X:B)_\sigma$ $\quad \sigma = \sum_x p_x |x\rangle\langle x| \otimes N(\rho_x)$

$= 0$ iff $\forall p_x, \rho_x$, σ = product state

iff $\forall \rho_x$, $N(\rho_x)$ indep of x

iff N trivial: Bob can simulate N without Alice



③ Will prove later $C(N)$ continuous (in N).

④ Is $\chi^{(1)}(N) = \chi^{(n)}(N)$?

Yes for some special channels

Proven in 2008 by Hastings $\exists$ counterexample.

(Will discuss more).

⑤ How to find $\chi^{(1)}(N)$?

Properties of the optimal ensemble (the $\{p_x, \rho_x\}$ in the max of $\chi^{(1)}(N)$) :

Uhlmann 9701014 , Schumacher & Westmoreland 9912122

① ρ_x's can be chosen pure

② d^2 pure states are sufficient, where d = output dim of N.

Pf of ① : elementary.

② : Careful use of Caratheodory's thm.

eg 1 N_p = d-dim depolarising channel.

$$N_p(\rho) = (1-p)\rho + p\frac{I}{d}$$

For any pure state $|\psi\rangle$,

$N_p(|\psi\rangle\langle\psi|) = (1-p)|\psi\rangle\langle\psi| + p\frac{I}{d}$

has spectrum $(1-p+\frac{p}{d}, \underbrace{\frac{p}{d}, \cdots, \frac{p}{d}}_{d-1 \text{ times}})$

} easy to diagonalize

$\therefore S(N_p(|\psi\rangle\langle\psi|)) = -(1-p+\frac{p}{d})\log(1-p+\frac{p}{d}) - \frac{d-1}{d}p\log\frac{p}{d} = S$

$\circledast$ independent of $|\psi\rangle$.

$\therefore$ Any ensemble max $S(\underbrace{\sum_x p_x N(|\psi_x\rangle\langle\psi_x|)}_{\frac{I}{d}})$ will do

choose $p_x = \frac{1}{d}$, $|\psi_x\rangle = |x\rangle$,

$\therefore \chi^{(1)}(N_p) = \log d - S$

Cor: $d=2$, $\chi^{(1)}(N_p) = 1 - H(\frac{p}{2})$ = capacity of classical BSC w.p. flipping $= \frac{p}{2}$.

In general, if ① $\mathcal{N}$ unital (ie $\mathcal{N}(I) = I$)

② each $|\psi_i\rangle$ minimizes $S(\mathcal{N}(|\psi_i\rangle\langle\psi_i|)) =: s$

③ $\frac{I}{d} \in \text{conv}(|\psi_i\rangle)$ (ie $\exists p_i$ s.t. $\sum_i p_i |\psi_i\rangle\langle\psi_i| = \frac{I}{d}$)

conv hull

then $\{p_i, |\psi_i\rangle\}$ optimal and

$\chi^{(1)}(\mathcal{N}) = \log d - s$.

Ex. find $\chi^{(1)}(\mathcal{N})$ for $\mathcal{N}(\rho) = 0.8\rho + 0.15 X\rho X + 0.05 Z\rho Z$.

Ex: find $\chi^{(1)}(E_p)$ for E_p = d-dim erasure channel with erasure prob p, compare to classical case. Note E_p not unital.

Properties (cta) -

③ The most distinguishable inputs need not be optimal.

ie nonortho ρ_x's may be needed.

- Fuchs PRL 79 (1997) 1162 found the first channel that requires non orthogonal optimal ensemble
- Amplitude damping channel shares the same property (9912122).

<u>Amplitude damping channel:</u>

$$N(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger$$

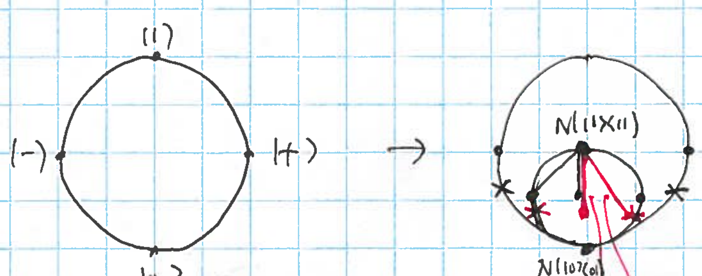
$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$$

$$a|0\rangle + b|1\rangle \xrightarrow{A_0} a|0\rangle + \sqrt{1-\gamma}\, b|1\rangle$$

$$a|0\rangle + b|1\rangle \xrightarrow{A_1} \sqrt{\gamma}\, b|0\rangle$$

If $|0\rangle, |1\rangle$ are energy eigenstates,
ground excited

then AD represents de-excitation.



distance from original should be small for the average, large for individual.

$\gamma = \frac{1}{2}$

- Restricting to $|\psi_0\rangle$ & $|\psi_1\rangle$, max @ $p_0 = p_1 = \frac{1}{2}$, $\chi = 0.4567$.
- Instead, $\chi = 0.4717$ for some $|\psi_0'\rangle$, $|\psi_1'\rangle$ with $\langle\psi_0'|\psi_1'\rangle \approx \cos 80^\circ$.

How hard is it to calculate or estimate $\chi^{(1)}(N)$?

Beigi & Shor 0707.2090:

Let $C \in \mathbb{R}$. To decide whether $\chi^{(1)}(N) > C$ or $< C - \varepsilon$ for $\varepsilon = \frac{1}{\text{poly}(d)}$ (d = input dim) is NP complete.

NB The above does not imply hardness of estimating $\chi(N^{\otimes n})$.

more structure