Lec 9

Goal: Classical capacity of quantum channels.

Intermediate goal: classical capacity of Q boxes ← aka CQ channels

A "Q-box" is specified by $\{\rho_x\}_{x \in \Omega}$.

If Alice inputs x, then Bob gets ρ_x:

$$x \Rightarrow \boxed{Q} \xrightarrow{\rho_x}$$

Obs 1: Similar to an ensemble, but no distribution imposed on x

Obs 2: If Bob applies measurement $\mathcal{B}$, $(Q, \mathcal{B})$ reduces to a classical channel:

$$x \Rightarrow \boxed{Q} \xrightarrow{\rho_x} \boxed{\mathcal{B}} \Rightarrow y$$

$pr(y|x)$ well defined.

$\therefore$ Asymptotic capacity of $(Q, \mathcal{B}) = \max_{p(x)} I(X:Y)$.

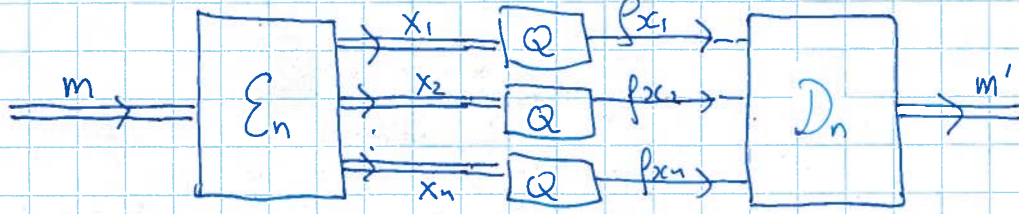
$Q \quad \geq \max_{p(x)} I_{acc}(\{p_x, \rho_x\})$.

↑ choose $\mathcal{B}$ optimal.

Can we do better? What is not general above?

(Bob).

General protocol:

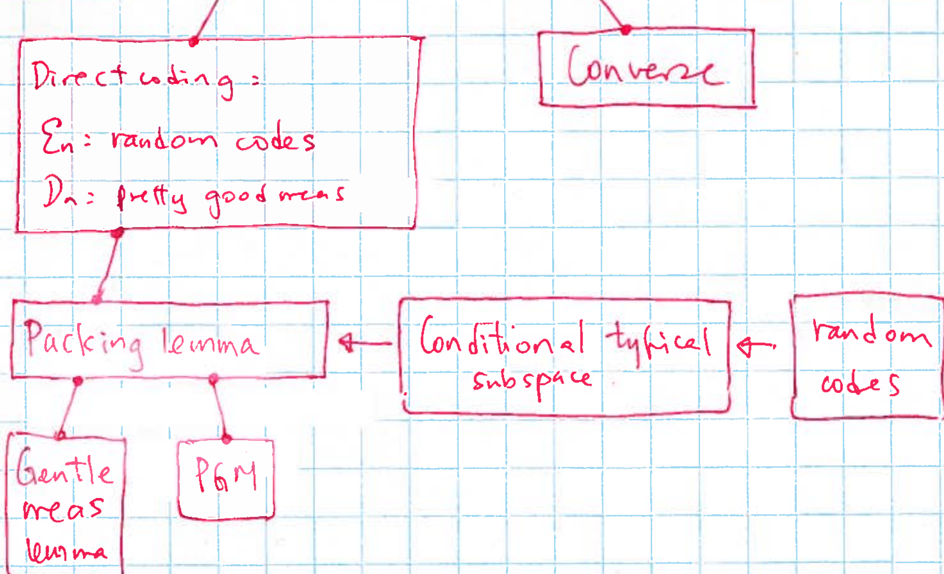


Alice inputs classical thing

★ Bob has to output the estimate of m. WLOG, he applies D_n on all q. states.

The notions of (M, n) code C_n, $P_e(m)$, $\mathbb{E} P_e(C_n)$, $P_e(C_n)$ achievable rate, and capacity carry through.
$\hat{C}(Q)$.

$$\boxed{\text{Thm: } C(Q) = \max_{p_x} \chi(\{p_x, \rho_x\}).}$$



① Gentle measurement lemma (Winter IEEE TIT 45(7) P2481-2485 1999)

Let $\rho \geq 0$, $\mathrm{tr}(\rho) \leq 1$, $0 \leq E \leq I$

If $\mathrm{tr}(\rho E) \geq 1-\eta$

then $\| E^{\frac{1}{2}} \rho E^{\frac{1}{2}} - \rho \|_{tr} \leq \sqrt{3\eta}$

Pf (after 3 pages of trace distance & fidelity).

Some useful measures of proximity:

Let $\mu, \nu \geq 0$, same dim.

Then • $D(\mu,\nu) \equiv \|\mu-\nu\|_{tr} := \frac{1}{2} tr|\mu-\nu| = \frac{1}{2}\|\mu-\nu\|_1$ ↙ Schatten 1-norm

• $F(\mu,\nu) := tr\sqrt{\mu^{\frac{1}{2}}\nu\mu^{\frac{1}{2}}} = \max_{|s\rangle,|t\rangle} |\langle s|t\rangle|$

Uhlmann's theorem

purifications of μ & ν.

We consider general $\mu, \nu \geq 0$.

In Nielsen & Chuang, $tr(\mu) = tr(\nu) = 1$.

Useful properties in this case:

① Monotonicity under q. ops $\mathcal{E}$

$$\|\mu-\nu\|_{tr} \geq \|\mathcal{E}(\mu)-\mathcal{E}(\nu)\|_{tr}$$
$$F(\mu,\nu) \leq F(\mathcal{E}(\mu),\mathcal{E}(\nu))$$

} Distinguishability cannot increase

② If μ,ν pure then $\|\mu-\nu\|_{tr} = \sqrt{1-F(\mu,\nu)^2}$

In general: $1-F(\mu,\nu) \leq \|\mu-\nu\|_{tr} \leq \sqrt{1-F(\mu,\nu)^2}$

See P404 Eq (9.22)
p404 Thm 9.2
p414 Thm 9.6
p415-416 Sec 9.2.3

But we want to consider the general $\mu,\nu \geq 0$ case.

Here are the statements & proofs in the general setting:

- Instead of (9.22): $\|\mu-\nu\|_{tr} = \max_{0\le P\le I} \operatorname{tr} P(\mu-\nu)$

- Simply note that in general, $\|\mu-\nu\|_1 = \max_{-I\le B\le I} \operatorname{Tr} B(\mu-\nu)$

- Thm 9.2 (monotonicity) still holds with a diff proof:

$$\|\mathcal{E}(\mu)-\mathcal{E}(\nu)\|_1$$

$= \operatorname{Tr} B^*(\mathcal{E}(\mu)-\mathcal{E}(\nu))$ where B^* maximizes $\operatorname{tr} B(\mathcal{E}(\mu)-\mathcal{E}(\nu))$

$= \operatorname{Tr} \mathcal{E}^+(B^*)\cdot(\mu-\nu)$ where $\mathcal{E}^+$ is the adjoint map

If $\mathcal{E}(\rho) = \sum_k A_k \rho A_k^+$

then $\mathcal{E}^+(\rho) = \sum_k A_k^+ \rho A_k$.

We have $\operatorname{tr} A\, \mathcal{E}(B) = \operatorname{tr} \mathcal{E}^+(A)\cdot B$

Also, $\mathcal{E}$ CP $\Leftrightarrow$ $\mathcal{E}^+$ CP

$\mathcal{E}$ TP $\Leftrightarrow$ $\mathcal{E}^+(I) = I$.

So $-I \le \mathcal{E}^+(-I) \le \mathcal{E}^+(B^*) \le \mathcal{E}^+(I) \le I$

$\le \|\mu-\nu\|_1$.

- For fidelity, Uhlmann's thm & monotonicity for $\operatorname{tr}\mu = \operatorname{tr}\nu = 1$ case implies that for the general $\mu,\nu\ge 0$ case, since the normalization can be inserted / removed throughout the proofs.

- If μ, ν pure but not necessarily has trace 1,

then, $\|\mu - \nu\|_{tr} \neq \sqrt{1 - f(\mu,\nu)^2}$ in general.

- If μ is pure, say $\mu = \alpha |a\rangle\langle a|$, $|a\rangle$ unit vector,

then $F(\mu,\nu) = \sqrt{tr(\mu\nu)}$ ($\star$).

$$\begin{aligned} Pf: \ F(\mu,\nu) &= tr\sqrt{\mu^{\frac12}\nu\mu^{\frac12}} \\ &= tr\sqrt{\sqrt{\alpha}\,|a\rangle\langle a|\,\nu\,|a\rangle\langle a|\,\sqrt{\alpha}} \\ &= \sqrt{\alpha}\sqrt{\langle a|\nu|a\rangle} \\ &= \sqrt{tr(\mu\nu)}. \end{aligned}$$

- If μ pure, $tr\,\mu = 1$, $tr\,\nu \leq 1$, then $\sqrt{1 - f(\mu,\nu)^2} \geq \|\mu - \nu\|_{tr}^2$

Pf: Let $\nu = \sum_j \lambda_j \nu_j$ where ν_j pure, trace 1

$\lambda_j > 0$

$\sum_j \lambda_j < 1$

$$\begin{aligned} \text{Then } \ &1 - f(\mu,\nu)^2 \\ &\overset{\star}{=} 1 - tr(\mu\nu) \\ &= 1 - \sum_j \lambda_j (tr\,\mu\nu_j) \\ &\overset{\star}{=} 1 - \sum_j \lambda_j f(\mu,\nu_j)^2 \end{aligned}$$

use NC Sec 9.2.3

$$\begin{aligned} &= 1 - \sum_j \lambda_j \left[1 - \|\mu - \nu_j\|_{tr}^2\right] \\ &= 1 - \sum_j \lambda_j + \sum_j \lambda_j \|\mu - \nu_j\|_{tr}^2 \end{aligned}$$

triangle ineq & convexity of x^2

$$\geq 0 + \|\mu - \sum_j \lambda_j \nu_j\|_{tr}^2 = \|\mu - \nu\|_{tr}^2$$

Pf of Gentle measurement lemma.

Let $|\psi\rangle_{RS}$ be any purification of ρ_S.

$\| E^{\frac{1}{2}} \rho E^{\frac{1}{2}} - \rho \|_{tr}$

$\overset{①}{\leq} \| I \otimes E^{\frac{1}{2}} |\psi\rangle\langle\psi| I \otimes E^{\frac{1}{2}} - |\psi\rangle\langle\psi| \|_{tr}$

$\overset{②}{\leq} \left[1 - |\langle\psi| I \otimes E^{\frac{1}{2}} |\psi\rangle|^2 \right]^{\frac{1}{2}}$

$\overset{③}{\leq} \left[1 - |\langle\psi| I \otimes E |\psi\rangle|^2 \right]^{\frac{1}{2}}$

$\overset{④}{\leq} \left[1 - tr(\rho E)^2 \right]^{\frac{1}{2}}$

$\leq \sqrt{2\eta + \eta^2} \leq \sqrt{3\eta}$

For ③: $E \leq E^{\frac{1}{2}} \quad \because E \leq I$

$I \otimes E \leq I \otimes E^{\frac{1}{2}}$

$\underbrace{\langle\psi| I \otimes E |\psi\rangle \leq \langle\psi| I \otimes E^{\frac{1}{2}} |\psi\rangle}$

both nonnegative

For ①: monotonicity in general case

for ②, $\| \|_{tr}^2 \leq 1 - F^2$

when μ pure, trace 1

ν trace ≤ 1.

For ④: Let $|\psi\rangle = \sum_i \sqrt{\lambda_i} |i\rangle_R |e_i\rangle_S$

st $\rho = \sum_i \lambda_i |e_i\rangle\langle e_i|_S$

Then $\langle\psi| I \otimes E |\psi\rangle$

$= \sum_i \sqrt{\lambda_i} \underbrace{\langle i|_R \langle e_i|_S \; I \otimes E \sum_j \sqrt{\lambda_j} |j\rangle_R} |e_j\rangle_S$

$= \sum_i \sqrt{\lambda_i} \langle e_i | E | e_i \rangle = tr(E\rho)$

② Pretty good measurement [Belavkin 75]

Let $\rho_1, \dots \rho_k$ be states in $\mathrm{Pos}(\mathbb{C}^d)$

The PGM has POVM:

- $M_i = \Lambda^{-\frac{1}{2}} \rho_i \Lambda^{-\frac{1}{2}}$ for $i = 1, \dots k$
- $M_{k+1} = I - \sum_{i=1}^{k} M_i$

Here $\Lambda = \sum_{i=1}^{k} \rho_i$ (positive semidef).

$= \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$ ($\lambda_j > 0$, $|\phi_j\rangle$ orthonormal)

$\Lambda^{-\frac{1}{2}} = \sum_j \lambda_j^{-\frac{1}{2}} |\phi_j\rangle\langle\phi_j|$.

ⓐ Checking it is a POVM:

- Let $\Pi = \sum_j |\phi_j\rangle\langle\phi_j|$ ← I − Null space of Λ
- Then $\sum_{i=1}^{k} M_i = \Lambda^{-\frac{1}{2}} \Lambda \Lambda^{-\frac{1}{2}} = \Pi$

$\therefore$ $M_{k+1} = I - \Pi \geq 0$, also $M_i \geq 0$ $\forall i = 1, \dots k$.

ⓑ Optimality discussion – out of scope.