Two more operational meanings for $SL(n)$:

1. Consider the following task.
   Referee prepares $|y\rangle^m_{RA}$, gives $A_1 A_2 \ldots A_n$ to Alice and she should transmit $A_1 \ldots A_n$ to Bob preserving correlation. This time, correlation is entanglement with $R_1 \ldots R_n$.

   Let $\rho = \text{Tr}_R 14X4|_{RA}$.

   Once again, transmitting the $d$-typical space of $f^{\otimes n}$ achieves the goal with rate $R \approx S(\rho)$. See Preskill last part of Sec 10.3.

   The resulting state in this case is (easiest to think in Schmidt bases).

   $$\rho_{\text{out}} = I \otimes I \otimes 14X4|_{14X4}^{\otimes n} + \text{Tr} [I \otimes I \otimes (I-\Pi_S) 14X4|_{14X4}^{\otimes n}] \cdot \text{EKR}$$

   $$\| \rho_{\text{out}} - 14X4|_{14X4}^{\otimes n} \|_1 \leq \frac{1}{2} \text{tr} (\Pi_S \rho^{\otimes n}) \leq \epsilon$$

   $$\| \rho_{\text{out}} - 14X4|_{14X4}^{\otimes n} \|_1 \leq 2 \int \left[ 1 - \left( \frac{4|\langle y|^{\otimes n} I \otimes \Pi_S 14|y\rangle^{\otimes n} R_{RA}^{\otimes n} \rangle}{R_{RA}^{\otimes n}} \right)^2 + \epsilon \right]$$

   $$\leq 2 \frac{1}{2} \int \left[ 1 - \frac{4|\langle y|^{\otimes n} I \otimes \Pi_S (y)|y\rangle^{\otimes n} R_{RA}^{\otimes n} \rangle}{R_{RA}^{\otimes n}} + \epsilon \right]$$

   $$\leq 2 \frac{1}{2} \epsilon + \epsilon.$$
Description of (4) known to Alice & Bob throughout

2. Entanglement concentration & dilution. Bennett-Barnstein-Popescu-Schumacher 95/11/30

Two related questions:

(a) How many qubits are required to prepare \( |4\rangle_{AB} \) with high fidelity?

(b) How many qubits can be extracted from \( |4\rangle_{AB} \) with high fidelity?

We allow arbitrary back & forth classical communication between A & B and local operations (LOCC) but nothing else.

Answer: \( \text{nS(Tr}_A |4\rangle\langle 4|_A \text{)} for both questions.} \)

(i) The fact there is an answer for either question is a surprise.

For pure states shared among 3 or more parties, the questions cannot even be well formulated (no equivalence of the sort).

(ii) For bipartite mixed states, answer for (a) \( \neq \) answer for (b) in general. So there is no unique quantification of entanglement.

(iii) For bipartite pure states, because of the surprisingly nice answers to (a) & (b), \( S(\text{Tr}_A |4\rangle\langle 4|_A \text{)} \) is a useful operational way to quantify entanglement in (4).

(iv) Method for (a) is called entanglement dilution concentration.

(v) Even more surprising: Conc requires NO classical communication.

Dilute requires \( \text{O(n)} \) qubits (and \( \text{O}(\text{ln}|n|) \) inf)opies).

Harrow-Lo 02, Hayden-Winter 02, First asked by Lo-Popescu 99/02/045.
(a) For dilution the protocol to distribute $1\text{E}_{\text{na}}$

using $\simeq n S(\text{tra} \, 1/2 \times 1)$ qubits can be converted

to one using $\simeq n S(\text{tra} 
1/2 \times 1)$ qubits $+$ $2\log_2\text{tra} \,(1/2 \times 1)$ qubits

due to teleportation. This already proves the claim, though

unnecessarily too many qubits are used.

(b) For concatenation: See A2.

Idea: Alice & Bob can apply local unitaries to turn

$|\Psi\rangle_{AB}$ into the Schmidt form

$|\Psi\rangle_{AB} = \sum_{x^n \in \mathbb{T}_n} \sqrt{p(x^n)} \, |x^n\rangle \langle x^n|.$

The tempting method is for both Alice & Bob to

measure their $n$ system w/ POVM \{ $\mathbb{T}_n$, $I-\mathbb{T}_n$ \}.

They get, with high prob $\leq \sum_{x^n \in \mathbb{T}_n} \sqrt{p(x^n)} \, |x^n\rangle \langle x^n|.$

"roughly" equal.

BUT it is not easy to lower bound the fidelity

of the above state & the maximally entangled state

$\sum_{x^n \in \mathbb{T}_n} \, |x^n\rangle \langle x^n| \; \frac{1}{|\mathbb{T}_n|^{1/2}}.$

For detail, see Preskill 10.4.
A good method is for both Alice & Bob to make much finer measurement (resulting state closer to MFE but has fewer Schmidt term).

Ref: For $X$ with sample space $S = \{1, 2, \ldots, m\}$,
where $X_1, X_2, \ldots, X_n$ is in the type class $(t_1, t_2, \ldots, t_m)$
where $t_k = \# x_i$ is equal to $k$.

Eq. $M = 4$, $n = 10$

1 4 2 3 3 2 1 2 4 3 is in the type class $(2, 3, 3, 2)$.

Eq. A coin tosses, any n-bit outcome with
- Hamming weight $k$ is in the type class $(n-k, k)$.

All $x^n$'s in the same type class are exactly equiprobable.

To concentrate entanglement, Alice & Bob independently measure the type class. They always get the same outcome.

The best measurement state is exactly maximally entangled but with a probabilistic amount of entanglement.

You will show in AZ the expected amount is

$S(tr_{14} 14 \times 14) = n - O(n)$

for $14^2 \propto 2^n$ but idea generalizes to arbitrary $14$.

as long as # type classes is small enough (as long as $k$ large enough).
The dilution protocol discussed in class:

Given: \( n \) \((S(p)+q)\) events between Alice & Bob, LOCAL

Want: \( (\psi)_{AB} \)\(^n\)

* Both Alice & Bob know what \( \psi \) is