What about q- analogues?

Let A B be two quantum systems

ρ (or ρ_{AB}) the state on AB

$S(AB) = S(\rho)$, $S(A) = S(\mathrm{tr}_B \rho)$, $S(B) = S(\mathrm{tr}_A \rho)$

↑ Add subscript to indicate the state if there is ambiguity.

Def Conditional entropy.

$$S(A|B) = S(AB) - S(B)$$

← Defined by the chain rule instead.

[NB: An initiation expression, no operational meaning analogous to classical case. Some insights eventually obtained 2005.]

* Chain Rule holds by definition.

Def Quantum relative entropy.

$$S(\rho\|\sigma) := \operatorname{tr}\rho[\log\rho - \log\sigma]$$

eg $\operatorname{tr}\rho\log\sigma$.

$\sigma = U D U^\dagger$

$\log\sigma = U \log D U^\dagger$

$\operatorname{tr}(\rho\, U \log D U^\dagger)$

Thm [Klein's ineq]:

$S(\rho\|\sigma) \geq 0$.

"=" iff $\rho = \sigma$.

Pf. NC [511-513].

Elementary. + $D(p\|q) \geq 0$.

Also used in hypo testing in Q setting.

Def [Quantum mutual info] :

$S(A:B) := S(A) + S(B) - S(AB)$:)

↑ Analogue of H4.

Sym.

Thm ← Analogue of def.

$$S(A:B) = S(\rho \,\|\, \rho_A \otimes \rho_B)$$

Pf:

Check if this is familiar.

$$S(\rho \,\|\, \rho_A \otimes \rho_B) = \operatorname{tr} \rho \log \rho - \operatorname{tr} \rho [\log \rho_A \otimes \rho_B]$$

$$= -S(AB) - \operatorname{tr} \rho (\log \rho_A \otimes I) - \operatorname{tr} \rho (I \otimes \log \rho_B)$$

$$= -S(AB) + S(A) + S(B)$$

Cor $S(A:B) \geq 0$, "=" iff $\rho = \rho_A \otimes \rho_B$.

Eg $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $S(A:B) = 1 + 1 - 0 = 2$

$S(A) \quad S(B) \quad S(AB)$

$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle|i\rangle$, $|i\rangle\langle i|$, $S(A) = S(B) = H(\lambda)$, $S(A:B) = 2H(\lambda)$.

$= \sum_i \lambda_i \log \lambda_i$

$S(AB) \geq 0$

Eg

$\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$, $S(A:B) = 1 + 1 - 1 = 1$

$S(A) \quad S(B) \quad S(AB)$

Properties of S:

(S0) Invariance:

- $S(\rho) = S(U\rho U^\dagger)$ for U unitary
- $S(A:B)_\rho = S(A:B)_{U_A\otimes U_B\,\rho\,U_A^\dagger\otimes U_B^\dagger}$

(S1) Range:

$0 \le S(\rho) \le \log d$ where $d = \dim(\mathrm{supp}(\rho))$

"$=$" iff $\rho = |\psi\rangle\langle\psi|$ "$=$" iff $\rho = \frac{I}{d}$

(~~S2~~) $S(A|B) = S(AB) - S(B)$ ← not a convex combination of $S(\rho_k)$'s.

Many possibilities: Say $\dim A = \dim B = d$.

- If $\rho = |\Phi\rangle\langle\Phi|$, $|\Phi\rangle = \frac{1}{\sqrt{d}}\left(|00\rangle + \cdots + |(d-1)(d-1)\rangle\right)$

  then $S(AB) = 0$, $S(B) = \log d$, $S(A|B) = -\log d$. !

- No particular characterization for $S(A|B) = 0$. ← Skip. except neg info.

- If $\rho = \frac{I}{d}\otimes\frac{I}{d}$,

  then $S(A|B) = \log d$.

(~~S3~~) $S(AB) \geqq S(A)$.

Will see later (Cor of S9 ~~S10~~) $[S(A|B) \le S(A)]$

(S4) Def of $S(A:B)$

Replace by $S(A:B) = S(\rho \| \rho_A \otimes \rho_B)$

(S5) Subadditivity. (SA)

$S(AB) \leq S(A) + S(B)$.

(By non neg of $S(A:B)$.

(S6) Conditioning reduces entropy. ← ?? meaning

$S(A|B) \leq S(A)$

[Claim ⇔ SA)

(S7). Concavity [Mixing Q states ⇒ entropy]

$$\sum_{k=1}^{t} \lambda_k S(\tau_k) \leq S\left(\sum_{k=1}^{t} \lambda_k \tau_k\right)$$

$\lambda_k \geq 0$, $\sum_k \lambda_k = 1$

$\tau_1, \tau_2, \ldots, \tau_t$ possible states on the same Q sys.

Pf: Consider $\rho = \sum_k \lambda_k |k\rangle\langle k|_A \otimes \tau_k{}_B$. (S10)

$S(AB) = S(\rho) = H(\{\lambda_k\}) + \sum_k \lambda_k S(\tau_k)$ ← (S6)!

$S(A) = H(\{\lambda_k\})$.

$S(B) = S\left(\sum_{k=1}^{t} \lambda_k \tau_k\right)$.

By SA: $S(AB) \leq S(A) + S(B)$.

$\Rightarrow \sum_k \lambda_k S(\tau_k) \leq S\left(\sum_{k=1}^{t} \lambda_k \tau_k\right)$.

(S8) Strong subadditivity (SSA) (also HP)

For any ρ_{ABC}:

$$S(C) + S(ABC) \leq S(AC) + S(BC).$$

NB 1 = "C" is special above. (of course, you can choose any of the sys to be "C")

NB 2 If $\dim C = 1$, above reduces to SA

$$0 + S(AB) \leq S(A) + S(B).$$

← A quick way to remember SSA. add "C" to all 4 terms.

NB 3 Holds for H & S.

Important equivalent statements:

(S8.1) Conditioning reduces conditional entropy.

$$S(A|BC) \leq S(A|C)$$

(S8.2) Conditional mutual information is non-negative

$$S(A:B|C) := S(A|C) - S(A|BC) \geq 0$$

(S8.3) Monotonicity of QMI wrt discarding

$$S(A:C) \leq S(A:BC)$$

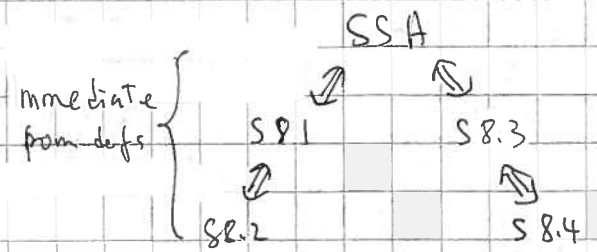
(S8.4) [Monotonicity of QMI] wrt local Q ops.

$$S(A:B) \leq S(A:B)$$

$\mathcal{I} \otimes \mathcal{E}(\rho)$ — $\mathcal{E}$ = CPmap on B; $\mathcal{I}$ = Id on A; ρ on AB

- Pf of SA = NC 11.4.1.

- of equivalences:



Pf [SSA ⇔ S8.3]:

S8.3 ⇔ $S(A \cup B) - S(A \cap C) \geq 0$

⇔ $S(A) + S(BC) - S(ABC) - [S(A) + S(C) - S(AC)] \geq 0$.

⇔ $S(BC) + S(AC) \geq S(C) + S(ABC)$

⇔ SSA.

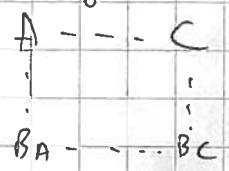
Pf [§8.3 ⇐ §8.4]

① Discarding is a TCP map.

② Discarding even a joint sys $B = B_A B_C$ is a composition of 2 local discards.

∴ 8.4 ⇒ 8.3

state on all 4 sys can be arbitrary



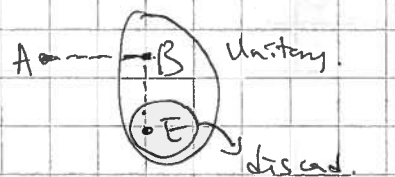
Pf [§8.3 ⇒ §8.4]

Any TCP map $\mathcal{E}$ can be realized by.

① Attaching ancilla ϵ in pure state (keeps QMI inv)

② Applying unitary on BE (when $\mathcal{E}$ local, so U ∴ QMI inv)

③ Discarding E' (QMI nonincreasing). from 8.3



∴ QMI nonincreasing under $\mathcal{I}_A \otimes \mathcal{E}_B$.

(8.9) Araki-Lieb inequality:

$$S(AB) \geq S(A) - S(B)$$

Equiv: $S(AB) \geq |S(A) - S(B)|$
[By sym of A & B]

(Pf) (1) Go to church of larger Hilbert space

$\exists$ sys C, pure state $|\psi\rangle$ on ABC
s.t. $\mathrm{tr}_C |\psi\rangle\langle\psi| = \rho_{AB}$.

We say "C purifies AB".

(2) By purity of ABC:

$$S(AB) = S(C), \quad S(A) = S(BC).$$

(3) $$S(B) + S(AB) \underset{(2)}{=} S(B) + S(C)$$

$$\underset{(SA)}{\geq} S(BC) \underset{(2)}{=} S(A)$$

Cor 1: $|S(AB) - S(A)| \leq S(B)$.

Cor 2: $|S(A:BC) - S(A:C)| \leq 2S(B)$.

} Will see other measures violating such "stg".

ie Adding or removing a system B changes vN entropy & QMI at most by $S(B)$ & $2S(B)$, however B is correlated with the existing system (S).

Pf: [Cor 1] AL: $S(A) - S(AB) \leq S(B)$.

SA: $S(AB) - S(A) \leq S(B)$.

Pf: [Cor 2] $|S(A:BC) - S(A:C)|$ $\overset{(\geq)}{\sim}$ non negative

$= |[S(BC) - S(ABC)] - [S(C) - S(AC)]|$ [2 $S(A)$ cancels out].

$= |[S(BC) - S(C)] + [S(AC) - S(ABC)]|$

$\leq |S(BC) - S(C)| + |S(AC) - S(ABC)|$

$\leq S(B) + S(B)$ ← Cor 1.

(S10). Entropy of classical-quantum systems:

Let $\rho_{XQ} = \sum_x p(x) \underbrace{|x\rangle\langle x|}_{X} \otimes \rho_x$

Then $S(XQ) = \underbrace{H(X)}_{\text{also } S(X).} + \sum_x p(x) S(\rho_x)$

} Pf: elementary ? A3 ?

NB: $S(Q|X) := S(XQ) - S(X) = \sum_x p(x) S(\rho_x)$

resumes classical interpretation when conditioning on a classical system.

(S11) QMI of classical-quantum systems:

ρ_{XQ} as above.

$S(X:Q) = S\left(\sum_x p_x \rho_x\right) - \sum_x p_x S(\rho_x) =: \chi(\{p(x), \rho_x\})$.

Holevo info of ensemble $\{p(x), \rho_x\}$.

Pf: $S(X:Q) = S(X) + S(Q) - S(XQ)$.

$\overset{(S10)}{=} S(X) + S(Q) - H(X) - \sum_x p(x) S(\rho_x)$.

$= S\left(\sum_x p_x \rho_x\right) - \sum_x p(x) S(\rho_x)$.

Other properties eg. cty of $S(\rho)$ as fcn of ρ - later.