Optimality of SD 2 vs TP:

SD: 1 qbit + 1 ebit \geq 2 cbits
TP: 2 cbits + 1 ebit \geq 1 qbit

If ent is free, SD & TP are inverses of one another & useful bit: QC = \frac{1}{2} CE.

Thm 1 (Optimality of TP)
If \( 2n < \alpha \) cbits + \( \eta \beta \) ebits \geq n qbits for some \( \beta \)
Then \( \alpha > 1 \).

Thm 2 (Optimality of SD)
If \( n \) qbit + \( n \beta \) ebits \geq 2n cbits for some \( \beta \)
Then \( \alpha > 1 \).

If even with unlimited ebits, the communication costs of SD & TP are optimal since no protocol can achieve the same communication task with less.
If (then 1):

Suppose a protocol $TP'$ exists that consumes $2n$ ebits and $n/\beta$ qubits to transmit $n$ qubits.

Now produce those $2n$ ebits by $SD$, consuming in turns $n/\beta$ qubits and $n$ ebits. Call the resulting protocol $TP''$:

So $TP''$ consumes $n/\beta + n$ ebits and $n$ qubits to generate $n$ qubits. By (C3), $n/\beta > n - 1$.

Summary with resource inequalities (RI):

By $SP$: $n$ qubits + $n/\beta$ ebits $\geq 2n$ ebits.

If $2n$ ebits + $n/\beta$ ebits > $n$ qubits then $n$ qubits + $n/\beta$ ebits + $n$ ebits $> n$ qubits.

$n > n - 1$.

Substitution of RE of $SP$ into that of $TP'$ requires $SD$ to be composable.
PF (Theorem 2):

If there is a protocol $SD'$ consuming $n_d$ qubits and $n_B$ ebits to transmit 2nd cbits

then we use 2nd cbits & nd ebits to perform TP
to supply those nd qubits.

In terms of RI:

$SD'$: $n_d$ qubits + $n_B$ ebits $\geq 2n$ cbits

$\forall$ TP

2nd cbits + nd ebits.

$SD''$: $2n$ 2nd cbits + $(n_d + n_B)$ ebits $> n$ cbits

$l_d > 1$ by (C).
Recall: 2 qubits $\rightarrow$ 1 qbit when entanglement is free.

What happens when entanglement is charged?

Recall: qbit = $(x)_A \rightarrow (x)_B$ for any basis $\{|i\}\_B$.

Isometric extensions:

- cbit = $(x)_A \rightarrow (x)\_B$.
- Cbit = $|0\_A\>_B \rightarrow |0\_B\>_A$.
- Cbit = $|1\_A\>_B$.

Coherent classical communication (Harrow 2003).

\begin{itemize}
  \item No one else has a copy.
  \item Alice must retain a copy.
  \item Erect must have a copy.
\end{itemize}

Why such a strange form of communication?

- Occurs naturally if classical comm is sent via coherent methods (SD, Unitary 2-way channels, ...)
- Allows efficient conversions between protocols.
- Completes the understanding of SD & TP.
Resource inequalities concerning qubits:

1. 1 qubit > 1 cbit
2. 1 qubit > 1 cbit
3. 1 cbit > 1 ebit
4. |qubit + 1 ebit| > 2 cbits (SD) \{ inverses of one another \}
5. |qubit + 2 ebits| ≤ 2 cbits + 1 ebit \(\text{TP}^{c0}\)  

Previously:

4. |qubit + 1 ebit| > \[2 \text{ cbits}\] (SD)
5. |qubit + [2 \text{ cbits}]| ≤ \[2 \text{ cbits}\] + 1 ebits. (TP)
PF (1) : $|x\rangle_A \rightarrow |x\rangle_A |\chi\rangle_B \rightarrow (|x\rangle \chi\rangle_B,$

"achieves 1 cbit"

USE 1 cbit

\[ \text{partial trace \ of } q_A \text{ (local)} \]

\[ \text{partial trace \ of } q_A \]

\[ \text{measure \ of } q_A \rightarrow |x\rangle_A |\chi\rangle_B |\chi\rangle_B \]

PF (2) : $|x\rangle_A \rightarrow |x\rangle_A |\chi\rangle_A \rightarrow (|x\rangle \chi\rangle_A |\chi\rangle_A$,

"achieves 1 cbit"

\[ \text{attach \ ancilla \ (local)} \]

\[ \text{local \ unitary} \]

\[ \text{USK \ (qubit \ on } A) \]

PF (3) : $\frac{1}{\sqrt{2}} \sum_{x=0}^{1} |x\rangle_A \rightarrow \frac{1}{\sqrt{2}} \sum_{x=0}^{1} |x\rangle_A |\chi\rangle_B$

"uses 1 cbit"

\[ \text{achieves \ 1 cbit} \]
Pf ④: We denote the conversion between the Bell basis and the computational basis by $U$:

$$
\begin{align*}
|x\rangle_U & \quad \mapsto \quad |(\sigma_3 \otimes I)|_{x+1}^A \\
U^{-1} & \quad \mapsto \quad \begin{cases} 
\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) & 
\end{cases}
\end{align*}
$$

for $x = 0, 1, 2, 3$. The following transformation achieves the claimed RI:

$$
\begin{align*}
|x\rangle_A \quad & \quad \left( \begin{array}{c} |00\rangle + |11\rangle \\
\frac{1}{\sqrt{2}} 
\end{array} \right)_{A \times B} \\
\text{uses 1 qubit} & \\
\text{achieves} & \\
2 \text{qubits} & \\
\text{achieves} & \\
\text{conditioned on } A \text{ being } x, \text{ apply } \sigma_3 \text{ to } A_x. \quad \text{(local unitary on } A A_x) \\
& \\
|x\rangle_A \quad & \quad \left( \begin{array}{c} |00\rangle + |11\rangle \\
\frac{1}{\sqrt{2}} 
\end{array} \right)_{A \times B} \\
\text{uses 1 qubit to send } A_x \text{ from Alice to Bob} & \\
\downarrow & \\
|x\rangle_A \quad & \quad \left( \begin{array}{c} |00\rangle + |11\rangle \\
\frac{1}{\sqrt{2}} 
\end{array} \right)_{B_1 B} \\
\text{apply } U^{-1} \text{ to } B_1 B. \quad (\text{unitary}) & \\
\downarrow & \\
|x\rangle_A \quad & \quad (\text{some}) \quad (B_1 B) 
\end{align*}
$$
Original SD:

Keeping everything coherent:

Simply put: SD is coherent, so if Alice keeps a coherent copy without "looking" (measuring, giving info to Bob) and similarly for Bob, the 2 doits become 2 cobits.
Let $TP^{\phi}$ be the protocol obtained from replacing the 3 qubits by 2 qubits in $TP$.

Verifying the math:

The $X$th Bell basis state:

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} \sum_{x=0}^{3} |x\rangle_A \otimes |x\rangle_B
\]

(Proved in les)

\[
1\text{bit}
\]

Apply $U_1$ on $A$, output $A_2$ has 4 dim, local:

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} \sum_{x=0}^{3} |x\rangle_A \otimes |x\rangle_B
\]

2 qubits taking $A_2$ to $A_2 B_2$

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} \sum_{x=0}^{3} |x\rangle_A \otimes |x\rangle_B
\]

Conditioned on $B_2$ being $x$, apply $U_2$ to $B$

Local unitary on $B_2 B$

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} \sum_{x=0}^{3} |x\rangle_A \otimes |x\rangle_B
\]

No more dependence on $x$.
Together:

\[ T^{\text{co}} : 1 \text{ebit} + 2 \text{cobits} \geq 2 \text{ebits} + 1 \text{gbit} \]

If lebit can be borrowed and returned, net effect:

\[ 2 \text{cobits} > 1 \text{lebit} + 1 \text{gbit} \]

the reverse of SD.

NB: Asymptotically, \[ 2 \text{cobits} = 1 \text{lebit} + 1 \text{gbit} \]
Extension of RI to bipartite quantum gates:

Consider

\[ A \begin{array}{c} U \\ B \end{array} \]

Such a gate can be nonlocal, transmitting quantum or classical data in either direction (forward from Alice to Bob or backwards from Bob to Alice) and generate entanglement.

E.g.: \( U = CNOT = 10 \otimes I_B + 11 \otimes X_A \otimes 0_B \)

- If Bob prepares \( |0_B \rangle \),
  
  then \( CNOT_{AB} |0_C \otimes 0_B \rangle = |0_C \rangle_A |0_B \rangle_B \)

  \[ \therefore CNOT_{AB} \rightarrow 1 \text{ qubit} \quad \text{ (from Alice to Bob)} \]

- Since

\[ A \begin{array}{c} \overline{\neg} \\ B \end{array} = \begin{array}{c} H \\ 1 \\ 0 \end{array} \]

  free local operations

\[ CNOT_{AB} \supset CNOT_{BA} \rightarrow 1 \text{ qubit} \quad \text{ (from Bob to Alice)} \]

- \( 1 \text{ qubit} \rightarrow \emptyset \): \( CNOT_{AB} \rightarrow 1 \text{ qubit} \)
Thin (Harrow 03):

1 CNOT + 1 ebit = 1 qbit_{(s)} + 1 qbit_{(c)}

1 SWAP = 1 qbit_{(c)} + 1 qbit_{(s)}

Cor: 1 CNOT $\not\Rightarrow$ 1 qbit.
Cor: 2 CNOT = 1 SWAP.

See Harrow 03 and/or Assignment?